Research Statement

Oleksandr Misiats

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1 Overview

I am an applied mathematician with background in Analysis, Calculus of Variations, Ordinary and Partial Differential equations, Stochastic Analysis and Homogenization. In particular, I study the qualitative behavior of nonlinear PDEs of elliptic and parabolic types, as well as variational problems associated with them. I have been developing mathematical methods that enhance our understanding of physical phenomena, such as formation of singularities in superconductors, microstructures in alloys, curvature-dependent evolutions or long-time behavior of systems influenced by random perturbations.

My PhD thesis was devoted to the analysis of variational problems related to the Ginzburg-Landau model of superconductivity and corresponding nonlinear elliptic equations. At the present time I am a Courant Instructor working on modeling microstructures in martensitic phase transitions using minimization techniques for nonconvex singularly perturbed energy functionals. Several of my works are on the long time behavior and asymptotics of stochastic reaction-diffusion equations. In addition, I worked on numerical discretizations for the nonlinear Richards equation and on the problems related to modeling curvature driven flows.

Calculus of Variations and Pattern Formation

Crumples in a sheet of paper, wrinkles on curtains, mixtures of phases in alloys, and Abrikosov lattices in superconductors are examples patterns in materials. My research goal is to understand the formation of such patterns from the point of view of energy minimization, which is is one of the challenges of modern Calculus of Variations.

The objects of my research are nonconvex energy minimization problems, regularized by higher order terms. Such models are common in materials science. The Ginzburg Landau model of superconductivity is one example of such models. This model was suggested by the physicists Landau and Ginzburg in the 1950s as a phenomenological description of superconductivity. The discovery of patterns of vortices (or defects) in superconductors led to the 2003 Nobel Prize in Physics, awarded to Ginzburg, Abrikosov and Leggett. In the papers [16] and [17] (with L. Berlyand and V.Rybalko) we establish the existence of minimizers with vortices, as well as, in certain cases, we find the locations of the vortices of these minimizers. The papers [18] and [19] (with P. Mironescu and M. Dos Santos) are devoted to the analysis of composite superconductors. In these works, described in a greater detail in Section 2, we are able to obtain the homogenized description of the superconductor with a large number of asymptotically small inclusions of a weaker superconductor. In certain regimes we also determine which of these inclusions contain vortices.

Another prominent example of a Calculus of Variations model which accurately captures the formation of experimentally observable patterns is modelling of martensitic phase transformations. A comprehensive description of microstructure formation in martensites and the associated shape-memory effect, can be found in the monograph [1] by K. Bhattacharya. A number of physical experiments indicate, that if two different pure phases of martensite are present at the opposite sides of a rectangular sample, the transition between these phases can take the form of a zigzag wall. In our work we consider nonconvex minimization problems that capture, we believe, the essential physics of this phenomenon. One of my projects at the Courant Institute (with R. Kohn and S. Muller) deals with the scalar elasticity model of phase transitions with two distinct phases of martensite present at the opposite sides of a rectangle. My second project (with R. Kohn) is devoted to the vectorial (3D elastic) version of this problem. In both cases we establish the global energy scaling laws, and show that the experimentally observed zigzag patterns provide the optimal energy scaling. In addition, in the scalar model we were able to obtain the results on the local distribution of the minimal energy, which is a fairly rare finding for variational problems of this type. These results are described in a greater detail in Section 3.

Stochastic Reaction Diffusion equations

Nonlinear stochastic reaction-diffusion equations describe physical and biological processes including heat explosion, tumor growth and the evolution of biological species in random environment. In these models one of the important questions is whether the quantity of interest (e.g. the tumor, the population etc.) stays bounded or continues to grow as time elapses. The answer to this question is far from trivial, as illustrated in the following example. Consider the equation

$$u_t = \Delta u + \sin(u) + F. \tag{1}$$

Here $x \in \mathbb{R}^d$, and $F \in \mathbb{R}$ is a fixed parameter. We are interested in the long time behavior of solutions of (1), specifically, in the existence of bounded solutions as $t \to \infty$. Clearly, if $|F| \leq 1$, bounded solutions exist, e.g. $u(t,x) \equiv -\arcsin F$. Conversely, if $|F| > 1$, no solution of (1) is bounded. However, even a small random perturbation of the right hand side of (1)

$$u_t = \Delta u + \sin(u) + F + \varepsilon \dot{W}(t), \varepsilon > 0 \tag{2}$$

completely changes the picture, since explicit bounded solutions are no longer available. To the best of our knowledge, the question whether one can find a solution of (2) satisfying $\sup_{t\geq 0} E\|u(t)\|^2 < \infty$ in some suitable norm, is still open for any $F \in \mathbb{R}$ and $\varepsilon > 0$. Despite the fact that currently we have not established the existence of a bounded solution of (2), the Section 4 describes a large class of stochastic reaction diffusion equations, for which bounded solutions exist. In addition, in certain cases we are able to show the uniqueness of a stationary solution to the stochastic reaction-diffusion equation. Establishing the existence and uniqueness of stationary solution is also important since it is a crucial step in showing the ergodic behavior of the corresponding system. We further investigated the asymptotic behavior of the stationary solutions under both regular and singular perturbations of homogenization type.

2 Ginzburg-Landau model (PhD thesis)

My PhD thesis was devoted to variational problems and PDEs related to the *Ginzburg-Landau superconductivity theory.* These problems are very interesting from a mathematical standpoint. In the process of their rigorous analysis we developed a number of novel mathematical concepts and methods, which may be useful in the analysis of other mathematical models. Moreover, these variational problems were motivated by physics and contribute to a deeper understanding of certain physical notions, such as *vortices*, which are an inherent feature of superconductivity.

Superconductivity is the complete loss of electrical resistance that occurs in certain materials below characteristic temperature. The Ginzburg-Landau (GL) model was introduced in 1950s as a phenomenological description of superconductivity. The success of the GL theory in predicting complex, physically observable phenomena in superconductivity led to the 2003 Nobel Prize in Physics. At the core of this theory is the **GL energy functional (GLF)**:

$$GL[u,A] := \frac{1}{2}\int_{\Omega} |\mathrm{curl}A - H|^2 + \frac{1}{2}\int_{G} (|(\nabla - iA)u|^2 + \frac{\lambda}{8}\int_{G}(1-|u|^2)^2, \quad G \subset \mathbb{R}^n,\ n = 2,3. \tag{3}$$

Here Ω is the smallest simply connected domain containing G, A is the vector potential of the induced magnetic field, H is the applied magnetic field, u is a complex-valued order parameter introduced so that $|u|^2$ is proportional to the density of superconducting particles (Cooper pairs), and λ is the Ginzburg-Landau parameter, which depends on the material. The standard normalization implies $|u| \leq 1$, with $|u| = 1$ corresponding to an ideal superconductor and $|u| = 0$ to a normal conductor. Isolated zeros of u around which the vector field u has a nonzero integer winding number (degree) are called *vortices*. Understanding the vortex structure of GLF minimizers (solutions of the GL PDE) is crucial, since vortices determine electromagnetic properties of superconductors that are important for practical applications (e.g., resistance).

In part I of my thesis (the works [16], [17] and [21]) we established the existence of minimizers (both global and local) of the Ginzburg-Landau energy functional (3), in which the external magnetic field H (which is the physical source of vortices) is replaced with the degree boundary conditions. In other words, we considered the minimization problem for (3) with $H = 0$ and $G \subset \mathbb{R}^2$ in the class

$$\mathcal{J}_d := \{(u,A) \in H^1(G,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2), |u| = 1 \text{ on } \partial G, deg(u,\partial G) = d > 0.\} \tag{4}$$

Here $deg(u,\partial G) := \frac{1}{2\pi}\int_{\partial G} u \times u_\tau ds$ is the winding number, which gives the lower bound on the number of zeros of an $H^1(G)$ - regular function. The function class $\mathcal{J}_d$ is a connected component in the space $\{(u,A) \in H^1(G) \times H^1(\Omega), |u| = 1 \text{ on } \partial G\}$. However, this class is not weakly-H^1 closed, thus the existence of minimizers for (3) in $\mathcal{J}_d$ is a highly nontrivial question.

The work [16] (with L. Berlyand and V.Rybalko) deals with Ginzburg-Landau energy functional in simply connected domains. We established the existence of minimizers with vortices (zeros) for certain values of the Ginzburg-Landau parameter (which are lower than the critical value $\lambda_{cr} = 1$). We also observed that the vortices of these minimizers were inside a fixed compact set (strictly inside the domain) for λ close to 1. On the contrary, in doubly connected domains [17] we observed a different picture. We established the existence of minimizers with *near boundary* vortices, which converge to the boundary of the domain as λ approaches λ_{cr}. Furthermore, we found the limiting location of the vortex on the boundary. The main ingredient of the proof was the energy expansion, obtained through sharp upper and lower bounds. As typical for problems of this type, the lower bound was the key challenge of this work.

The work [21] focused on the local minimization problem for simplified Ginzburg-Landau functional (obtained from (3) via setting both $A = 0$ and $H = 0$) in doubly connected domain $G = \Omega \setminus \omega$:

$$E_\varepsilon[u] := \frac{1}{2}\int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 dx \tag{5}$$

This minimization problem is a subject to "semi-stiff" boundary conditions $|u| = 1$ and prescribed degrees p and q on the outer $\partial\Omega$ and inner $\partial\omega$ boundaries respectively. It is not difficult to see that due to the lack of compactess of the boundary conditions, the global minimizers of this minimization problem seize to exist. In order to find local minimizers, following the work of L. Berlyand and V.Rybalko [20], we additionally prescribe the degree in the bulk (approximate bulk degree), which, in the simplest case of a radially symmetric domain $G = B_R(0) \setminus B_r(0)$ is given by

$$abdeg(u, G) := \frac{1}{\log(R/r)} \int_r^R \left(\frac{1}{2\pi} \int_{|x|=\xi} u \times u_\tau ds \right) \frac{d\xi}{\xi}$$

If u_ε is chosen such that $E_\varepsilon[u_\varepsilon] < \Lambda$, where $\Lambda > 0$ is a fixed constant, then for sufficiently small ε, the quantity $abdeg(u_\varepsilon, G)$ is close to some integer $d \in \mathbb{Z}$. The work [20] established the *sufficient* conditions on the existence of Ginzburg-Landau minimizers. Specifically, it is shown that if $d > 0$ ($d < 0$) is such that $d \geq \max\{p, q\}$ ($d \leq \min\{p, q\}$), then the minimizers exist in the class

$$J^d_{p,q} := \{u \in H^1(G), |u| = 1 \text{ on } \partial G, deg(u, \partial\Omega) = p, deg(u, \partial\omega) = q, abdeg(u, G) \in (d - 1/2, d + 1/2)\}.$$

My work [21] complements the result of [20] and provides the necessary conditions for the existence of minimizers. Specifically, using sharp upper and lower bounds, I show that if $d > 0$ ($d < 0$) and $d \leq \min\{p, q\}$ ($d \geq \max\{p, q\}$) with $p \neq q$, the minimizers in $J^d_{p,q}$ do not exist. The question of the existence of minimizers in $J^d_{p,q}$ with $p < d < q$ requires a different approach. This question is currently open.

Part II (the works [18] and [19]) was devoted to modeling composite superconductors. This work was motivated by physical models of vortex pinning (i.e., fixing the positions of vortices), which was done by introducing inclusions into a homogeneous superconductor. In the work [18] (in collaboration with P. Mironescu and M. Dos Santos) we consider Ginzburg-Landau type functional

$$E^a_\varepsilon[u] := \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - a^2_\delta)^2 \, dx \tag{6}$$

with the rapidly oscillating discontinuous pinning term $a_\delta(x)$, which takes values $0 < b < 1$ on a two-dimensional δ-periodic array of inclusions inside G, and value 1 otherwise. The starting point of our analysis is the strikingly simple decoupling strategy due to Lassoued and Mironescu [8]. Let U_ε be the unique scalar minimizer of (6) subject to the (scalar) boundary condition $U_\varepsilon \equiv 1$ on ∂G. It is not difficult to verify that $b \leq U_\varepsilon \leq 1$, with U_ε being close to a away from the inclusions' boundaries. Thus, without loss of generality we look for the minimizers of (6) in the form $u = U_\varepsilon v$, which leads to the following decomposition:

$$E_\varepsilon[u] = E_\varepsilon[U_\varepsilon] + F_\varepsilon[v], \tag{7}$$

where

$$F_\varepsilon[v] = \frac{1}{2} \int_G U^2_\varepsilon |\nabla v|^2 + \frac{1}{4\varepsilon^2} U^4_\varepsilon (|v|^2 - 1)^2 \, dx. \tag{8}$$

Despite the fact that the Euler-Lagrange equation for (8) is nonlinear, we are able to adapt the linear homogenization techniques to obtain homogenization limits for the minimizers of (8) depending on the relation between δ and ε (i.e. $\delta >> \varepsilon$, $\delta = \varepsilon$ and $\delta << \varepsilon$). The subsequent work [19] (with M. Dos Santos) was focused on locating the small inclusions with vortices. We showed that even the inclusions of negligibly small size $\delta \to 0$ capture the vortices of minimizers, provided $\frac{\ln \delta^3}{\ln \varepsilon} \to 0, \varepsilon \to 0$. Furthermore, we reduced the problem of finding the locations of the vortices of minimizer to a discrete minimization problem for a finite-dimensional functional.

3 Pattern formation in variational problems

My first project at Courant Institute (in collaboration with R.Kohn and S. Muller, preprint) deals with the minimization problem for the functional

$$I_\varepsilon[u] := \int_R (|u_x|^2 + \varepsilon |u_{yy}|) dx dy \to \min \tag{9}$$

Here $R = [0,1] \times \left[-\frac{1}{2}, \frac{1}{2}\right]$, and the minimization is performed in the class of functions

$$W := \{u \in H^1(R), |u_y| = 1, u(0,y) = y, u(1,y) = -y\}. \tag{10}$$

In other words, the admissible functions can have either $u_y = 1$ or $u_y = -1$, and thus u_{yy} is a Radon measure with finite mass. The functional (9) is a sum of elastic and surface energies, and can be used to model phase transitions in alloys, in which the regions with $u_y = 1$ and $u_y = -1$ correspond to two distinct phases of a martensite. The boundary conditions $u = \pm y$ at the left and right edges of the box respectively indicate that different phases are present at the boundaries. Some physical experiments, e.g. [2], [3], [4], suggest the formation of zigzag patterns, however the rigorous explanation of this phenomenon in the physics literature is missing.

The earlier work [23] studied the same functional with different boundary conditions. In particular, as shown in [23], the boundary condition $u = 0$ at the left side of the box R leads to the emergence of branched microstructures. While these microstructures have oscillations with the approximate period of $\varepsilon^{1/3}$ in the bulk of a sample, they have much finer scales closer to the left boundary, since the boundary condition is not compatible with either of the phases of the martensite. The energy in that case is of order $\varepsilon^{2/3}$ and is concentrated primarily near the left boundary. That is, for a.e. $\rho \in (0,1)$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} (u_x^2 + \varepsilon|u_{yy}|)(\rho, y)dy \geq C\varepsilon^{2/3}\rho^{-2/3}. \tag{11}$$

The subsequent work of Conti [9] showed that the minimizer, roughly speaking, has a self-similar microstructure near the boundary, and established local-in-x and local-in-y energy estimates for the minimizers. It is worth mentioning that, in contrast to establishing the global energy scaling laws, results on the local features of the minimizers for nonconvex variational problems are rare.

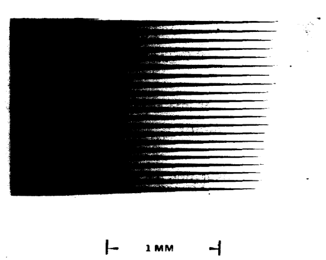

Figure 1: Experimentally observed zigzag wall, [3]

The minimization problem for (9) in (10), which is the object of our current research, is of somewhat different nature due to the presence of nontrivial relaxed energy. If we relax the constraint $|u_y| = \pm 1$ to $|u_y| \leq 1$, which corresponds to the convexification of the double-well potential $(u_y^2 - 1)^2$, the minimum of (9) in $W_0 := \{u \in H^1(R), |u_y| \leq 1, u(0,y) = y, u(1,y) = -y\}$ is attained by an explicit relaxed solution $u_*(x,y) = (2x-1)y$ with $I_\varepsilon[u_*] = \frac{1}{3}$. Direct integration shows that $I_\varepsilon[u_\varepsilon - u_*] = I_\varepsilon[u_\varepsilon] - I_\varepsilon[u_*]$. In contrast, the problem considered in [23] and [9] has $u_* = 0$ with $I_\varepsilon[u_*] = 0$. We start with the global scaling law.

Theorem 3.1. *Let u_ε be a minimizer of $I_\varepsilon[u]$ in (10). Then*

$$c_1\varepsilon^{2/3} \leq I_\varepsilon[u_\varepsilon - u_*] \leq c_2\varepsilon^{2/3} \text{ with } c_1 \approx 2.56 \text{ and } c_2 \approx 2.62.$$

The lower bound is proved using an anzatz-free argument. The upper bound is established via constructing a piecewise linear (in both x and y) test function, reminiscent of the experimentally observed zigzag construction. While at this point we cannot say that this zigzag test function is an actual minimizer, we can say that it shares some common features with the minimizer beyond providing the global energy scaling. In particular, we establish the local energy distribution.

Theorem 3.2. *Let u be a minimizer with $u - u_*$ satisfying periodic boundary conditions at $y = \pm\frac{1}{2}$. Denote $R_\rho = [0,\rho] \times [-\frac{1}{2}, \frac{1}{2}]$ with $\rho > 0$. Then*

$$\int_{R_\rho} (u - u_*)_x^2 + \varepsilon|u_{yy}|\, dx\, dy \leq C_1\varepsilon^{2/3}\rho|\ln(\rho)|^2 \tag{12}$$

and

$$\int_{-1/2}^{1/2} (u - u_*)^2(\rho, y)dy \leq C_2\rho^2|\ln(\rho)|^2. \tag{13}$$

Notice that the energy in this case behaves significantly different, as compared to the branching case of [23]. In particular, (12) indicates that the energy is distributed almost uniformly (modulo the logarithmic factor), as opposed to the branching case, when the energy blows up as we approach the left boundary of the sample. The zigzag test function satisfies both (12) and (13) without the logarithmic prefactors, which, we believe, can be removed. We also establish the dependence of the optimal energy on the vertical size of the domain via establishing the existence and the rate of convergence of the thermodynamic limit.

Theorem 3.3. *If u^N is the minimizer of*

$$I_\varepsilon^N[u] := \int_0^N \int_0^1 [(u-u^*)_x^2 + \varepsilon|u_{yy}|]dxdy, \tag{14}$$

then, for any $N \geq 1$, we have

$$\frac{1}{N} I_\varepsilon^N[u^N] = I_\varepsilon^1[u^1] + o(\varepsilon^{2/3}), \varepsilon \to 0, \tag{15}$$

and the limit of the left hand side as $N \to \infty$ exists. (Recall that, by Theorem 3.1, $I_\varepsilon^1[u^1] \sim \varepsilon^{2/3}$).

Clearly, had we known that $u - u_*$ was periodic, (15) would have read $\frac{1}{N} I_\varepsilon^N[u^N] \equiv I_\varepsilon^1[u^1]$, which holds for the zigzag test function.

Current work. My second project (with R.Kohn) is devoted to modeling the phase transitions using the 3D elastic analog of the scalar functional (9). This problem is motivated by the physical experiments in [5] and [6], where the zigzag transitions were observed experimentally under the mechanical deformation (bending) of a thin alloy over a cylinder. In our model we assume that a material has two preferred elastic strains

$$e_{23} = e_{32} = \pm 1; e_{ij} = 0 \text{ otherwise;}$$

where $e_{ij} := \frac{\partial_{x_i} u_j + \partial_{x_j} u_i}{2}$, $1 \leq i, j \leq 3$. We study the minimization problem

$$E_\varepsilon[u_1, u_2, u_3] := \int_{R_1} [e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{13}^2 + \varepsilon|\partial_{x_3} e_{23}|]dx \to \min \tag{16}$$

with $R_1 = [-1, 1]^3$, in the class

$$W = \{(u_1, u_2, u_3) : e_{23} = \pm 1, e_{23} = 1 \text{ at } x_1 = 1 \text{ and } e_{23} = -1 \text{ at } x_1 = -1.\}$$

The relaxed solution in this case becomes

$$\begin{cases} u_1^* = -x_2 x_3; \\ u_2^* = x_1 x_3; \\ u_3^* = x_1 x_2. \end{cases}$$

However, in contrast to the scalar case, $E_0[u_1^*, u_2^*, u_3^*] = 0$. Our main result is the global energy scaling law

$$c_1 \varepsilon^{2/3} \leq E_\varepsilon[u_1, u_2, u_3] \leq c_2 \varepsilon^{2/3}. \tag{17}$$

The above scaling result should also hold if the constraints $e_{23} = \pm 1$ at $x_1 = \pm 1$, which correspond to the presence of pure phases at the boundaries, are replaced with a volume fraction constraint, which would allow mixtures of phases at $x_1 = \pm 1$ with different volume fraction of two phases at each component of the boundary. The upper bound in (17) is obtained by the explicit zigzag construction, suggested by the experiment. To establish the lower bound, we use a convex duality argument. Since the constraint $e_{23} = \pm 1$ is nonconvex, the duality argument may be applied only to the relaxed problem. However, even the minimization of the relaxed problem in some, well chosen subdomain of R_1, with the strain constraint being present only on the boundary of this subdomain, turns out to be sufficient to establish the optimal energy scaling law.

Future work. The scalar model (9) and the vectorial model (16) are examples of linear elasticity models of martenisitic phase transitions. We next plan to extend our results to the nonlinear elasticity setting. S Conti et al [10] established that the branched microstuctures, reminiscent of the ones discovered in [23], provide the optimal scaling law in the nonlinear elasticity model with Dirichlet boundary conditions. In particular, the authors considered the minimization problem

$$E_\varepsilon[u] := \int_R [W(Du) + \varepsilon|D^2u|]dx \to \min, u \equiv Id \text{ on } \partial R.$$

Here

$$W(Du) := dist(Du, K), \text{ with } K = SO_2A_- \cup SO_2A_+,$$

where SO_2 is the group of rotations and $A_\pm$ are preferred martensitic states, e.g.

$$A_\pm = \begin{pmatrix} 1 & \pm\alpha \\ 0 & 1 \end{pmatrix} \quad \text{or } A_\pm = \begin{pmatrix} 1 & 0 \\ 0 & 1 \pm \alpha \end{pmatrix} \tag{18}$$

for $\alpha \in (0, 1)$. We plan to consider the analogous problem with the preferred states (18) prescribed at the right and left edges of R. Our conjecture is that in this case the zigzag patterns provide the optimal energy scaling law as well.

4 Long time behavior of stochastic reaction-diffusion equations

We consider a nonlinear stochastic equation of parabolic type, perturbed with an infinite dimensional Wiener process

$$du = (Au + f(x, u))dt + \sigma(x, u)dW(t, x) \tag{19}$$

Here $u \in H$, where H is a certain (real) Hilbert space (typically $L^2(\mathbb{R}^d)$ or weighted $L^2_\rho(\mathbb{R}^d)$, A is an elliptic operator on H, which generates a continuous semigroup $S(t)$, f and σ are real measurable functions, and $W(t, \cdot) \in H$ is a nuclear infinite dimensional Wiener process. We call the process $u = u(t, x, \omega)$ to be a mild solution of (19) with the initial condition u_0 s.t. $E\|u_0\|^2 < \infty$, if it is H-valued, measurable with respect to an appropriate filtration, and for all $t \geq 0$ satisfies

$$u(t) = S(t)u_0 + \int_0^t S(t-s)[f(u(s))]ds + \int_0^t S(t-s)\sigma(u(s))dW(s) \tag{20}$$

with probability 1.

Equations of this type model the behavior of various dynamical systems in physics and mathematical biology, such as electric potential on nerve cells in Hodgkin-Huxley model in neurophysiology etc. We are interested in the analysis of long-time behavior of solutions of (19), in particular, in the study of ergodic properties of the system (19). A solution of (19), which is a stationary process, defines an invariant measure for (19). Furthermore, as shown in [26, Theorem 3.2.6], the uniqueness of the invariant measure implies that the solution process is ergodic.

One of the crucial steps in establishing the existence of an invariant measure is the pioneering result of Krylov and Bogoliubov, which implies that if a solution of (19) satisfies the uniform boundedness condition

$$\sup_{t\geq 0} E\|u(t, x, \omega)\|^2_H < \infty, \tag{21}$$

then the invariant measure for (19) exists, provided the semigroup $S(t)$ satisfies certain compactness properties.

The behavior of solutions of (19) is significantly different in bounded and unbounded domains. The principal difference is that the semigroup $S(t)$ of an elliptic operator in a bounded domain G with homogeneous Dirichlet boundary conditions has an exponential contraction property

$$\|S(t)u\|_{L^2(G)} \leq Ce^{-\lambda_1 t}\|u\|_{L^2(G)}, \; u \in L^2(G), \tag{22}$$

where $\lambda_1 > 0$ is the first eigenvalue of A in G (subject to Dirichlet boundary condition $u = 0$ on ∂G). The estimate (22) implies the existence and uniqueness of the stationary solution for a large class of Lipschitz nonlinearities $f(x,u)$ and $\sigma(x,u)$, which, in turn, yields the existence and uniqueness of the invariant measure.

On the contrary, in unbounded domains, the validity of the estimate of type (22) heavily depends on the spectral properties of the operator A. In particular, this estimate does not hold for $A = \Delta$ in $\mathbb{R}^d$. Therefore, generally speaking, the dissipative properties of the diffusion operator in the entire space are often not sufficient for the existence of a stationary solution, and additional dissipative properties of the nonlinearity $f(x,u)$ are needed.

The question of the existence of invariant measures in unbounded domains with $A = \Delta$ was studied, e.g., in [28]. Loosely speaking, the key result of the work [28] states that there exists an invariant measure for (19) provided f satisfies the following dissipation condition

$$uf(u) \leq -ku^2 + c \tag{23}$$

for some $k > 0$ and $c \in \mathbb{R}$.

The work [14] established the existence of an invariant measure for (19) with $A = \Delta$ in $\mathbb{R}^d$ under a different conditions on f. In that paper, we show that the invariant measure for (19) exists if f satisfies the global bound:

$$|f(x,u)| \leq \varphi(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \forall u \in \mathbb{R}. \tag{24}$$

The case of a nonlocal nonlinearity $f(u)$, which appears, e.g., in the model of nonlocal consumption of resources [32], was also considered in [14].

In the subsequent work [12] we use a new approach to (19) which is based on the generalized Ito's formula. This approach allows us to expand the class of nonlinearities in the right hand side of (19). We show that the equation (19) has a stationary solution if $\sigma(x,u)$ is bounded and there exists $M > 0$ and non-negative $\eta(x) \in L^1(\mathbb{R}^d)$ such that for all $x \in \mathbb{R}^d$ we have

$$uf(x,u) \leq \eta(x) \text{ for } |u| > M \text{ and } x \in \mathbb{R}^d. \tag{25}$$

Notice that the condition (25) is a much weaker condition compared to (23). In this paper we also show that (19) has a bounded solution if $f(x,u)$ and $\sigma(x,u)$ are Lipschits functions in u, and the Lipschitz constant L has a certain rate of spatial decay.

The paper [13] was devoted to the asymptotic behavior of the stationary solutions, established in [14] and [12]. Given $\varepsilon > 0$, consider the equation

$$du = (\Delta u - \varepsilon u)dt + \sigma(x,u)dW(t,x), \ x \in \mathbb{R}^d. \tag{26}$$

The techniques that we used in [14] imply that the equation (26) has a unique stationary solution u_ε^*, since for every $\varepsilon > 0$ the semigroup for the operator $A_\varepsilon u := \Delta u - \varepsilon u$ has the exponential contraction property. This stationary solution can be obtained as a limit of a simple iteration scheme. On the other hand, the existence result for the stationary solution of the limiting equation

$$du = \Delta u dt + \sigma(x,u)dW(t,x), \ x \in \mathbb{R}^d \tag{27}$$

is rather abstract and not constructive [28]. Furthermore, we cannot say that the stationary solution of (27) is unique. We show that within the set of stationary solutions of (27) we can identify the unique solution u^* of (27), which is a limit of the stationary solutions of u_ε^* of (26). This way we provide the selection principle for the stationary solutions of (27). In addition, we can also efficiently approximate one of the stationary solutions of (27).

We also characterize the convergence of invariant measures in the homogenization limit. In particular, we show that

$$E\|u_\varepsilon(t) - u_0(t)\|_\rho^2 \to 0, \varepsilon \to 0.$$

where u_ε and u_0 are the corresponding unique stationary solutions of

$$du_\varepsilon = \left[div\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) - u_\varepsilon + f\left(\frac{x}{\varepsilon}, u_\varepsilon\right)\right]dt + \sigma\left(\frac{x}{\varepsilon}, u_\varepsilon\right)dW(t) \tag{28}$$

and

$$du_0 = [div(A_0 \nabla u_0) - u_0 + f_0(u_0)]\, dt + \sigma_0\,(u_0)\, dW(t). \tag{29}$$

Here $A(x)$ is a measurable, periodic and uniformly elliptic matrix, $f(x, u)$ and $\sigma(x, u)$ are uniformly bounded, continuous, periodic functions in x and Lipschits in u, f_0 and σ_0 are the corresponding averages of f and σ over the period cell Π, and A_0 is the homogenized matrix. Both equations are studied in the weighted space $L^2_\rho(\mathbb{R}^d)$ with $\rho(x) = e^{-\kappa|x|}$. Our analysis is based on the two-scale convergence approach [11] and on Nash-Aronson estimates.

Future work. I plan to extend the above homogenization result to the case of large reaction terms. In particular, the deterministic homogenization problem with large reaction term

$$\frac{du_\varepsilon}{dt} = div\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) + \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}, u_\varepsilon\right) \tag{30}$$

was considered in the work [33] by Allaire and Piatnitskii. In this work the authors derive the homogenized problem

$$\frac{du_0}{dt} = div(A_0 \nabla u_0) + F(u_0)\nabla u_0 + V(u_0), \tag{31}$$

where F and V are expressed via the solutions of the corresponding cell problems. Our preliminary calculations indicate that the approach suggested by the authors may be extended to the stochastic case as well. This way we would obtain the convergence of the corresponding stationary solutions in spirit of (4).

In addition, I would like to consider the long time behavior and the existence of invariant measures for stochastic primitive equations

$$\begin{cases} dv + (-\Delta v + v \cdot \nabla v + w\partial_z v + \nabla p)dt = f(v)dt + \sigma(v)dW; \\ \nabla \cdot v + \partial_z w = 0; \\ \partial_z p = 0 \end{cases} \tag{32}$$

Here the unknowns are the velocity field (v, w) and the pressure scalar p. This equation is a form of Navier-Stokes equation, used in climatology as a fundamental model for a number of geophysical flows. The results on the existence of invariant measures in the case $f = 0$, under some conditions on σ, were obtained in [35]. On the other hand, the work of M. Hieber [34] shows the existence of periodic solution in for certain deterministic nonlinearities. We plan to adapt techniques that we developed for the reaction-diffusion equations in this case, and show the existence of invariant measures for (32) in the case when both f and σ are different from 0.

5 Analysis of numerical discretizations for Richards' equation

My project with K.Lipnikov (Los Alamos National Laboratory) involved the analysis of numerical discretizations for the nonlinear Richards equation

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(K(x)a(u)(\nabla u + g\overrightarrow{e_1})\right) \tag{33}$$

The equations of type (33) are used in modeling water flow in multilayered unsaturated soils. In it, u stands for the capillary pressure, $K(x)a(u)$ is nonlinear hydraulic conductivity, constant g is gravity and x measures the depth. Finding an accurate numerical discretization for the equations of type (33) is a challenge even in 1D. The reason is that the solutions of (33) may have arbitrarily steep gradients, and thus standard numerical methods may lead to overshoots or undershoots (i.e. the loss of monotonicity), even on relatively fine grids. Thus, our goal was to develop a numerical discretization that would be monotone in the regions, where the analytical solution of (33) is monotone, i.e. does not have local extrema. Another computational challenge for Richards equation is lack of a closed form solution for general nonlinearities. In the work [29] we were able to derive an explicit solution for a certain class of discontinuous nonlinearities. The derivation of this explicit solution allowed us to test a number of numerical schemes for accuracy and convergence. We found a second-order accurate monotone numerical scheme for (33), and derived the necessary and sufficient conditions for monotonicity of numerical discretizations for the stationary solutions of (33).

6 Modeling discrete motion by mean curvature

My paper [15] (with N.K. Yip) addresses the convergence issues related to a space-time discrete thresholding scheme for motion by mean curvature.

The analysis of motion by mean curvature (in which the normal velocity of a moving manifold is given by its mean curvature) is an active area. Not only it is interesting in geometry in its own right, it also finds many applications in materials science and image processing. It is a prototype of a gradient flow with respect to the area functional. Due to the possibility of singularity formation and topological changes of the evolving surface, elaborate approaches need to be used. These include (i) varifold formulation, (ii) the viscosity solutions, and (iii) singularly perturbed reaction diffusion equation $u_t = \Delta u - \frac{1}{\varepsilon^2}W'(u)$, where $W(u) = (1-u^2)^2$.

The thresholding scheme is a particularly simple algorithm to capture the key feature of (iii). It is essentially a time splitting scheme. The first step is diffusion while the second step is thresholding to mimick the fast reaction due to the nonlinear term. The following simple thresholding scheme was suggested by Bence et.al [31]. Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain, and $\tau > 0$ be a small parameter. Let $\Omega_0 := \Omega$. For each $k \geq 0$ and $t \geq 0$ consider $w_k(x,t)$ s.t.

$$\begin{cases} \frac{\partial w_k}{\partial t}(x,t) = \Delta w_k(x,t) \text{ in } \mathbb{R}^2 \times (0,\tau) \\ w_k(x,0) = \chi_{\Omega_k}(x). \end{cases} \tag{34}$$

and set $\Omega_{k+1} := \{x \in \mathbb{R}^2 : w_k(x,t)|_{t=\tau} \geq \frac{1}{2}\}$. The sets $\Omega_k, k \geq 0$ determine the evolution of Ω at the moments of time $\{k\tau, k \geq 0\}$. As shown both in [22] and [24], this evolution approximates the motion by mean curvature in the limit $\tau \to 0$.

Now, for given $h > 0$ (grid size), we considered a semidiscrete version of (34). Let $\Omega_0^h := \{(m,n) \in \mathbb{Z}^2 : (mh, nh) \in \Omega\}$ and

$$\begin{cases} \partial_\tau w_k^{m,n}(t) = \frac{1}{h^2}[w_k^{m+1,n} + w_k^{m-1,n} + w_k^{m,n+1} + w_k^{m,n-1} - 4w_k^{m,n}] \\ w_k^{m,n}(0) = \chi_{\Omega_k^h} \end{cases} \tag{35}$$

Analogously to Bence's scheme [31], in [15] we define $\Omega_{k+1}^h := \{(m,n) \in \mathbb{Z}^2 : w_k^{m,n}(t)|_{t=\tau} \geq \frac{1}{2}\}$. However, in contrast to the continuum thresholding scheme (34), intricate pinning and depinning of the interface can be described with (35) in certain regimes. This is analogous to a gradient flow in a highly wiggling or oscillatory energy landscape. Our main result is the description of the evolution of Ω_k^h depending on the asymptotic relation between two small parameters h and τ, namely:

(i) *Subcritical case* $h << \tau$. We have the motion by mean curvature, similar to the continuum case of Osher et. al., i.e. the front motion does not notice the local minima of the energy surface.

(ii) *Supercritical case* $h >> \tau$. The domain Ω_0 does not evolve at all, i.e. $\Omega_k \equiv \Omega_0$, and the front gets stuck in a local minimum.

(iii) *Critical case* $\tau = \mu h$. The boundary points of Ω move with the velocity, different from mean curvature at this point. In particular, if a boundary point moves by $n_0 h$ (i.e. n_0 steps in the normal direction) at time τ, then n_0 is a function of the curvature κ, implicitly defined via the following relation

$$\sum_{k=1}^{n_0-1} \int_0^{\sqrt{\frac{2k}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx + \frac{1}{2}\int_0^{\sqrt{\frac{2n_0}{\mu\kappa}}} e^{-\frac{x^2}{4}}\,dx = \frac{1}{2}\int_{\sqrt{\frac{2n_0}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx + \sum_{k=n_0+1}^{\infty} \int_{\sqrt{\frac{2k}{\mu\kappa}}}^{\infty} e^{-\frac{x^2}{4}}\,dx \tag{36}$$

We also derive an analogous formula for the anisotropic motion in the case, when the front intersects the grid at an angle θ. To establish those results, we first obtain the discrete heat kernel representation for the solutions of (35)

$$w^{m,n}(t) = \sum_{i,j\in\mathbb{Z}^2} G_{m-i}\left(\frac{2t}{h^2}\right) G_{n-j}\left(\frac{2t}{h^2}\right) w^{i,j}(0),$$

$$G_n(\alpha) := e^{-\alpha} I_{|n|}(\alpha), \text{ where } I_n(\alpha) = \sum_{m=0}^{\infty} \frac{\left(\frac{\alpha}{2}\right)^{2m+n}}{m!(m+n)!} \text{ is a Modified Bessel function.} \tag{37}$$

We apply the Meissel formula for modified Bessel function to get uniform asymptotic expansions as $n \to \infty$ and $\alpha \to \infty$. We use those expansions to derive sharp lower and upper bounds for the sums of the discrete heat kernels and eventually obtain the desired results.

7 Ongoing work, future work and open problems

I plan to work in two areas: the pattern formation using the Calculus of Variation techniques, and the long time behavior of stochastic equations of reaction-diffusion type. My current and future directions in these areas are described in Sections 3 and 4 respectively.

In addition, there is a number of open questions that emerged from my research. The answer to some of these question is of a particular interest to me. One of such questions is on the long time behavior of a stochastic reaction-diffusion equation with oscillating potential, described in the overview. Another open question is on the existence of local Ginzburg-Landau minimizers in the class $J^d_{p,q}$ in the case, when $0 < p < d < q$. This question is described at the end of Section 3.

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