Near boundary vortices in a magnetic Ginzburg-Landau model: their locations via tight energy bounds

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Abstract

Given a bounded doubly connected domain $G \subset \mathbb{R}^2$, we consider a minimization problem for the Ginzburg-Landau energy functional when the order parameter is constrained to take $\mathbb{S}^1$-values on ∂G and have degrees zero and one on the inner and outer connected components of ∂G, correspondingly. We show that minimizers always exist for $0 < \lambda < 1$ and never exist for $\lambda \geq 1$, where λ is the coupling constant ($\sqrt{\lambda/2}$ is the Ginzburg-Landau parameter). When $\lambda \to 1-0$ minimizers develop vortices located near the boundary, this results in the limiting currents with δ-like singularities on the boundary. We identify the limiting positions of vortices (that correspond to the singularities of the limiting currents) by deriving tight upper and lower energy bounds. The key ingredient of our approach is the study of various terms in the Bogomol'nyi's representation of the energy functional.

Key words: PDEs with lack of compactness, Calculus of variations, Ginzburg-Landau model, Vortices
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1. Introduction

We study vortices located near the boundary (hereafter referred to as the near boundary vortices) that appear in 2D Ginzburg-Landau model when the order parameter is constrained to take $\mathbb{S}^1$-values on the boundary of a

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domain. Such a boundary condition models perfectly superconducting state of the system at the boundary. Following [5], we call this boundary condition along with the natural one, the semi-stiff boundary conditions (Dirichlet for the modulus of the order parameter and Neumann for the current, see details below). Mathematically, semi-stiff conditions can be regarded as a relaxation of $\mathbb{S}^1$-valued Dirichlet boundary data considered in the pioneering work [6] and pursued in [1], [17], [19] among others. In contrast to the Dirichlet boundary value problem, semi-stiff boundary conditions lead, in general, to ill posed variational and boundary value problems.

More specifically, given a bounded domain $G \subset \mathbb{R}^2$, we consider the problem of finding critical points of the Ginzburg-Landau free energy functional

$$F_\lambda[u, A] = \frac{1}{2}\int_G \left(|\nabla u - iAu|^2 + \frac{\lambda}{4}(|u|^2 - 1)^2\right) \mathrm{d}x \ + \frac{1}{2}\int_{\mathbb{R}^2} |\mathrm{curl} A|^2 \, \mathrm{d}x \quad (1.1)$$

in the space $(u, A) \in \mathcal{J} \times H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, where

$$\mathcal{J} = \{u \in H^1(G; \mathbb{C});\ |u| = 1 \text{ a.e. on } \partial G\}. \quad (1.2)$$

The unknowns in (1.1) are the map $u : G \to \mathbb{C}$ (order parameter) and the vector field $A : \mathbb{R}^2 \to \mathbb{R}^2$ (the potential of magnetic field); $\lambda > 0$ is a given coupling constant ($\sqrt{\lambda/2}$ is the Ginzburg-Landau parameter). As shown in [10] the space $\mathcal{J}$, endowed with the strong-H^1 topology, is not connected. Its connected components are obtained by prescribing the topological degree of u on components of the boundary ∂G. It is natural then to seek critical points of functional (1.1) by minimizing on the connected components of the space. However, the existence of minimizers of the latter minimization problems is nontrivial because of a possible lack of compactness of minimizing sequences. This is due to the fact that the degree on the boundary is not preserved in weakly-H^1 convergent sequences.

In the case of simply connected domain G the minimizers of (1.1) with prescribed degree on the boundary were studied in [10] for the special integrable (self-dual) case of the critical value $\lambda = 1$ of the coupling constant. Recently, in [4], this problem was considered for the full range of the parameter λ (where the elegant self-duality argument no longer applies). It was shown in [4] that

- minimizers with prescribed nonzero degree always exist for $0 < \lambda < 1$ and never exist for $\lambda > 1$ (for $\lambda = 1$ minimizers exist but there are also minimizing sequences that do not converge);

- in the limit $\lambda \to 1-0$ vortices of minimizers converge to certain inner points of the domain, these points maximize a finite dimensional functional.

In this work we consider the simplest case of multiply connected domain. Namely, we assume that $G = \Omega \setminus \bar{\omega}$, where Ω, ω are smooth bounded simply connected domains in $\mathbb{R}^2$, and $\bar{\omega} \subset \Omega$. We consider the subspace $\mathcal{J}_{01} \subset \mathcal{J}$ consisting of maps u whose topological degrees on $\partial\omega$ and $\partial\Omega$ are zero and one, correspondingly. Note that, by a simple topological consideration, every $u \in \mathcal{J}_{01}$ has at least one essential zero (in the Lebesgue sense). The variational problem we are interested in is

$$m(\lambda) = \inf\{F_\lambda[u, A];\ u \in \mathcal{J}_{01}, A \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)\}. \tag{1.3}$$

In this work we show that $m(\lambda)$ is always attained for $0 < \lambda < 1$ and never attained for $\lambda \geq 1$. The nonattainability of $m(1)$, which stands in sharp contrast to the case of simply connected domain, leads to a singular behavior of minimizers as $\lambda \to 1-0$. Namely, near boundary vortices appear, and their properties, primarily locations, are the main concern of this work.

Our principal result is

Theorem 1. *Let $0 < \lambda < 1$ and let (u^λ, A^λ) be minimizer of (1.1) in $\mathcal{J}_{01} \times H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$. Then we have, as $\lambda \to 1-0$*

(i) u^λ has exactly one zero (vortex) ξ^λ;

(ii) up to extracting a subsequence, $\xi^\lambda \to \xi^* \in \partial\Omega$ as $\lambda \to 1-0$ and ξ^* minimizes $|\partial V/\partial\nu|$ on $\partial\Omega$, where $\partial V/\partial\nu$ is the normal derivative of V and V is the unique solution of the (scalar) problem

$$\begin{cases} \Delta V = V \ \text{ in } G \\ V = 0 \ \text{ on } \partial\Omega, \ \text{ and } V = 1 \ \text{ on } \partial\omega; \end{cases} \tag{1.4}$$

(iii) the tangential component of the current $j^\lambda = (iu^\lambda, \nabla u^\lambda - iA^\lambda u^\lambda)$ on $\partial\Omega$ converges to $2\pi\delta_{\xi^*}$ in $\mathcal{D}'(\partial\Omega)$, where δ_{ξ^*} is the Dirac delta centered at ξ^*.

Remark 1. *In the course of the proof of Theorem 1 we show that (u^λ, A^λ) converges weakly in $H^1(G; \mathbb{C}) \times H^1(\tilde{G}; \mathbb{R}^2)$ (for every bounded domain $\tilde{G}$) to a limit (u, A) which is equivalent (modulo a gauge transformation) to a trivial minimizer ($u = \mathrm{const} \in \mathbb{S}^1, A = 0$). The singular behavior appears in the currents, as stated in (iii) of Theorem 1.*

Note that the singular behavior of minimizers is rather unusual. In particular, it is different from the one described in [8], where a related problem is studied in London limit of large λ. Along with the prescribed degree of the order parameter, a Dirichlet boundary condition for the tangential component of the current is imposed in [8]. This yields a well-posed variational problem for all $\lambda > 0$, moreover, vortices of minimizers converge to inner points described by a renormalized energy functional. The distinguishing feature of (1.3) is that the tangential component of the currents exhibits δ-like behavior on ∂G as $\lambda \to 1-0$, since vortices converge to the boundary points (unlike in [8]). The normal component of currents is always zero (insulating boundary condition), that is a natural boundary condition for (1.3).

For the simplified Ginzburg-Landau functional (obtained by setting $A = 0$ in (1.1)) minimizers with prescribed degrees were studied in [15], [3], [2], see also [16] for a related problem in another context. The results of these works suggest that when there is an energy reason or a topological reason for vortices to appear, minimizers do not exist. However, solutions with vortices of the corresponding semi-stiff problem (local minimizers in the space $\mathcal{J}$) do exist for multiply connected domains, as shown in [5](see also [11]). The vortices of the these solutions are located near the boundary and thus they are similar to that described in Theorem 1.

While the variational techniques developed in [5] (in particular, the lower and upper bounds) are sufficient to prove the existence of local minimizers with vortices, they do not allow one to determine the locations of vortices which is a key issue in the theory of Ginzburg-Landau type problems. For inner vortices the variational methods of [6] lead to a renormalized energy functional that captures limiting locations of that vortices. This approach, however, is not readily applicable to the near boundary vortices.

In this work we develop alternative techniques of tight upper and lower bounds for problem (1.3) that allow one to capture limiting locations of vortices on the boundary as $\lambda \to 1-0$. We emphasize that these limiting boundary vortices are seen in limiting currents rather than limiting order parameter (unlike inner vortices that have been extensively studied in the literature). The crucial point in our analysis is the following asymptotic (as $\lambda \to 1-0$) lower bound for the minimizing pair (u^λ, A^λ),

$$F_\lambda[u^\lambda, A^\lambda] \geq \pi + \frac{2\pi^2}{K_G}\delta^2 \Big|\frac{\partial V}{\partial \nu}(\xi^*)\Big|^2(1+o(1)) - \pi(1-\lambda)\delta^2|\log\delta|(1+o(1)) \quad (1.5)$$

where $\xi^* = \xi^*(\lambda)$ is the nearest point projection on $\partial\Omega$ of the unique zero (vortex) ξ^λ of u^λ, K_G is a positive constant (that depends on G only) and δ is the distance from ξ^λ to $\partial\Omega$ ($\delta = \delta(\lambda)$ tends to zero as $\lambda \to 1-0$). This bound is complimented by the matching upper bound of the same form, where $\xi^* \in \partial\Omega$ and (small) $\delta > 0$ are parameters (local coordinates of a point $\xi \in G$ near $\partial\Omega$). Therefore, we can minimize the right hand side of (1.5) first in δ to get the asymptotic relation $-\log\delta = \frac{2\pi}{(1-\lambda)K_G}\left|\frac{\partial V}{\partial\nu}(\xi^*)\right|^2(1+o(1))$, and then in ξ^* to show (ii) of Theorem 1. This yields also the following energy expansion $F_\lambda[u^\lambda, A^\lambda] = \pi - \exp\left(-\frac{4\pi}{(1-\lambda)K_G}M_G(1+o(1))\right)$, where $M_G = \min\{\left|\frac{\partial V}{\partial\nu}(\xi)\right|^2; \xi \in \partial\Omega\}$. Note that the problem of finding limiting locations of vortices is nonlocal in the sense that we must minimize $|\frac{\partial V}{\partial\nu}|$ on $\partial\Omega$, while $|\frac{\partial V}{\partial\nu}(\xi)|$ depends on the geometry of the entire domain G (not only local properties of the boundary $\partial\Omega$ at ξ).

The external magnetic field is zero in the energy functional (1.1) (only the induced magnetic field curl$A$ is present). We refer to [18] and references therein for the studies of models with nonzero external field.

This paper is organized as follows. Next section contains necessary preliminaries. In Section 3 we derive an upper energy bound in terms of solutions of a one parameter family of semilinear boundary value problems (3.3)-(3.4). On the basis of this upper bound, in Section 4, we establish the existence of minimizers of problem (1.3) for $0 < \lambda < 1$ (the approach there is similar to that of [3]). In Section 4 we also show the nonattainability of $m(\lambda)$ for $\lambda \geq 1$ by using the strong maximum principle and Hopf's lemma. Sections 5 and 6 constitute the core of this work. We show there the optimality of the upper energy bound for $\lambda \to 1-0$ by deriving the matching lower bound. To this end we perform an asymptotic decoupling of the Euler-Lagrange system for the minimizing pair (u^λ, A^λ) that leads to the study of a family of maps θ^λ with harmonic components, constant moduli on the connected components of ∂G, and satisfying the Cauchy-Riemann equations up to an error with controlled (small) L^p-norms (for $p = 2$ and $p < 2$). In Section 6 we prove a key lemma (see Lemma 3), which describes maps θ^λ versus their "projections" on a family of holomorphic maps with prescribed zeros. Section 7 describes vortices of minimizers and currents on the boundary. Finally, in Section 8 we use a linearization argument to get the explicit bounds of the form (1.5) and complete the proof of Theorem 1.

2. Preliminaries

In this paper we use the following notations and conventions:

- Every closed curve is counterclockwise oriented. For such a curve τ and ν stand for the unit tangent and unit normal vector vectors, respectively, that agree with the orientation ((ν,τ) is direct).

- The complex plane $\mathbb{C}$ is identified with $\mathbb{R}^2$, so that if $x, y \in \mathbb{C}$ then $(x,y) = \frac{1}{2}(x\bar{y} + y\bar{x})$ and $x \wedge y = \frac{i}{2}(x\bar{y} - y\bar{x})$ are the scalar and the wedge products, respectively.

- Given a fixed orthonormal frame (x_1, x_2) in $\mathbb{R}^2$, $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$ denote the classical Cauchy operators. For a scalar (real-valued) function f, $\nabla^{\perp} f$ is the vector field given by $\nabla^{\perp} f = (-\partial f/\partial x_2, \partial f/\partial x_1)$. For a vector field A, $\mathrm{curl}A = \partial A_2/\partial x_1 - \partial A_1/\partial x_2$.

- If $u \in H^{1/2}(\Gamma; \mathbb{S}^1)$ (where Γ is either $\partial\Omega$ or $\partial\omega$), then $\deg(u,\Gamma)$ is the topological degree (winding number) given by

$$\deg(u,\Gamma) = \frac{1}{2\pi}\int_{\Gamma} u \wedge \frac{\partial u}{\partial \tau}\mathrm{d}s,$$

  where the integral is understood via $H^{1/2}$-$H^{-1/2}$ duality.

- $B_r(y)$ denotes an open disk with the radius r and the center at y.

One of the main properties of the functional (1.1) is its invariance under gauge transformations $u \mapsto e^{i\phi}u$, $A \mapsto A + \nabla\phi$ (where $\phi \in H^2_{\mathrm{loc}}(\mathbb{R}^2)$). This allows us to reduce the study of (1.1) to the functional (still denoted $F_\lambda[u,A]$)

$$F_\lambda[u,A] = \frac{1}{2}\int_G \left(|\nabla u - iAu|^2 + \frac{\lambda}{4}(|u|^2-1)^2\right)\mathrm{d}x \; + \frac{1}{2}\int_\Omega |\mathrm{curl}A|^2\,\mathrm{d}x \qquad (2.1)$$

(see, e.g., [18]). Moreover, without loss of generality, we can assume that A is in the Coulomb gauge, i.e.

$$\begin{cases} \mathrm{div}A = 0 \text{ in } \Omega \\ A\cdot\nu = 0 \text{ on } \partial\Omega. \end{cases} \qquad (2.2)$$

Thus the minimization problem (1.3) can be equivalently restated as

$$m(\lambda) = \inf\{F_\lambda[u,A];\ u \in \mathcal{J}_{01}, A \in H^1(\Omega;\mathbb{R}^2) \text{ and } A \text{ satisfies (2.2)}\}. \quad (2.3)$$

Recall that

$$\mathcal{J}_{01} = \{u \in H^1(G;\mathbb{C});\ |u| = 1 \text{ a.e. on } \partial G, \deg(u,\partial\Omega) = 1, \deg(u,\partial\omega) = 0\}.$$

Critical points of $F_\lambda[u,A]$ in $\mathcal{J} \times H^1(\Omega;\mathbb{R}^2)$, in particular, minimizers of (2.3), are solutions of the system of Euler-Lagrange equations

$$-(\nabla - iA)^2 u + \frac{\lambda}{2} u(|u|^2 - 1) = 0 \text{ in } G \quad (2.4)$$

$$-\nabla^\perp h = \begin{cases} j \text{ in } G \\ 0 \text{ in } \omega, \end{cases} \quad (2.5)$$

where $h = \mathrm{curl} A$ is the magnetic field (scalar real-valued function in 2D), and

$$j = (iu, \nabla u - iAu)$$

is the current. Furthermore, $h \in H^1(\Omega)$ and the following boundary conditions are satisfied,

$$|u| = 1, j \cdot \nu = 0 \text{ on } \partial G,\ h = 0 \text{ on } \partial\Omega, \frac{\partial h}{\partial \tau} = 0 \text{ on } \partial\omega. \quad (2.6)$$

We assume that $\partial G \in C^\infty$, then we have $u \in C^\infty(\bar{G};\mathbb{C})$ and $A \in C^\infty(\bar{G};\mathbb{R}^2)$. This regularity property is established analogously [4]. We also have the pointwise inequality

$$|u| \leq 1 \text{ in } G,$$

which is a consequence of the maximum principle, since we have

$$\Delta|u|^2 = \lambda|u|^2(|u|^2 - 1) + 2|\nabla u - iAu|^2 \text{ in } G. \quad (2.7)$$

The following energy representation plays an important role in the analysis of problem (2.3) and it is valid for every $u \in \mathcal{J}_{01}$ and $A \in H^1(\Omega;\mathbb{R}^2)$,

$$F_\lambda[u,A] = \pi + F^+[u,A] + \frac{1}{2}\int_\omega |\mathrm{curl} A|^2 \mathrm{d}x - \frac{1-\lambda}{8}\int_G (|u|^2 - 1)^2 \mathrm{d}x, \quad (2.8)$$

where

$$F^+[u,A] = 2\int_G \Big|\frac{\partial u}{\partial \bar{z}} + \frac{A_2 - iA_1}{2} u\Big|^2 \mathrm{d}x + \frac{1}{2}\int_G \Big|\mathrm{curl} A + \frac{|u|^2 - 1}{2}\Big|^2 \mathrm{d}x. \quad (2.9)$$

This representation is due to a remarkable observation of Bogomol'nyi [9]. A detailed derivation of (2.8) can be found in [10].

3. Upper bound construction

To obtain an upper bound for $m(\lambda)$ we introduce a family of testing pairs $(u^{(\xi)}, A^{(\xi)}) \in \mathcal{J}_{01} \times H^1(\Omega;\mathbb{R}^2)$ that depends on the parameter $\xi \in G$ (the unique zero of $u^{(\xi)}$). We are seeking $u^{(\xi)}$ and $A^{(\xi)}$ in the form

$$u^{(\xi)} = \tilde{u}^{(\xi)}, \ A^{(\xi)} = \begin{cases} E^{(\xi)} + B^{+} \text{ in } G \\ B^{-} \text{ in } \omega, \end{cases} \tag{3.1}$$

with $(\tilde{u}^{(\xi)}, E^{(\xi)})$ minimizing $F^{+}[\tilde{u}, E]$ over $(\tilde{u}, E) \in \mathcal{J}_{01} \times H^1(G;\mathbb{R}^2)$ such that $\tilde{u}(\xi) = 0$. To simplify the notations we suppress the dependence of $B^{\pm}$ on the parameter ξ.

Clearly $F^{+}[\tilde{u}^{(\xi)}, E^{(\xi)}] \geq 0$ and the equality $F^{+}[\tilde{u}^{(\xi)}, E^{(\xi)}] = 0$ leads to the system of the first order partial differential equations,

$$\frac{\partial \tilde{u}^{(\xi)}}{\partial \bar{z}} + \frac{E_2^{(\xi)} - iE_1^{(\xi)}}{2}\tilde{u}^{(\xi)} = 0 \text{ and } \mathrm{curl} E^{(\xi)} + \frac{1}{2}(|\tilde{u}^{(\xi)}|^2 - 1) = 0 \text{ in } G. \tag{3.2}$$

The latter system is reduced, by Taubes' procedure (see [20]) of factorizing $\tilde{u}^{(\xi)}$ into the product of the holomorphic part $\gamma_\xi(z)$ and the factor $e^{\varphi_\xi/2}$, to the following single second-order equation for φ_ξ,

$$-\Delta\varphi_\xi + |\gamma_\xi(z)|^2 e^{\varphi_\xi} = 1 \text{ in } G. \tag{3.3}$$

In order to have $|\tilde{u}^{(\xi)}| = 1$ on ∂G, we supplement (3.3) with the boundary condition

$$\varphi_\xi = -2\log|\gamma_\xi(z)| \text{ on } \partial G. \tag{3.4}$$

We choose a special holomorphic map $\gamma_\xi \in H^1(G;\mathbb{C})$ that satisfies

$$\frac{\partial \gamma_\xi}{\partial \bar{z}} = 0 \text{ in } G; \ \gamma_\xi(\xi) = 0; \ \begin{cases} |\gamma_\xi| = 1 \text{ on } \partial\Omega, \ \deg(\gamma_\xi, \partial\Omega) = 1 \\ |\gamma_\xi| = \text{const on } \partial\omega, \ \deg(\gamma_\xi/|\gamma_\xi|, \partial\omega) = 0. \end{cases} \tag{3.5}$$

These conditions define γ_ξ uniquely, up to a constant factor of modulus one. Moreover, if we fix a conformal map $\mathcal{F}$ from Ω onto the unit disk $B_1(0)$, and set

$$a_\xi(z) = \frac{\mathcal{F}(z) - \mathcal{F}(\xi)}{1 - \overline{\mathcal{F}(\xi)}\mathcal{F}(z)}, \tag{3.6}$$

then $\sigma_\xi = \log|\gamma_\xi/a_\xi|$ is a (unique) harmonic in G function satisfying the boundary conditions $\sigma_\xi = 0$ on $\partial\Omega$, $\sigma_\xi = \text{const} - \log|a_\xi|$ on $\partial\omega$, and

$$\int_{\partial\omega} \frac{\partial\sigma_\xi}{\partial\nu}\,\mathrm{d}s = 0. \tag{3.7}$$

Thanks to the last condition, there exists a single valued harmonic conjugate $\psi_\xi \in C^\infty(\bar{G})$ ($\frac{\partial\psi_\xi}{\partial\bar{z}} = i\frac{\partial\sigma_\xi}{\partial\bar{z}}$) so that $\gamma_\xi = a_\xi \exp(\sigma_\xi + i\psi_\xi)$ satisfies (3.5). Next we set

$$\tilde{u}^{(\xi)} = \gamma_\xi e^{\varphi_\xi/2} \text{ and } E^{(\xi)} = -\frac{1}{2}\nabla^\perp\varphi_\xi. \tag{3.8}$$

It is shown in [10] (Theorem 4.3) that there is a unique solution $\varphi_\xi \in H^2(G)$ of the problem (3.3)-(3.4).

Next step is the construction of $B^\pm$ in (3.1). Using (2.8)-(3.2) and (3.8), we get

$$\begin{aligned} F_\lambda[u^{(\xi)}, A^{(\xi)}] = \pi &+ \frac{1}{2}\int_G \left(|B^+|^2 + (\mathrm{curl}B^+)^2\right)\mathrm{d}x + \frac{1}{2}\int_\omega |\mathrm{curl}B^-|^2\,\mathrm{d}x \\ &+ \frac{1}{2}\int_G \left((|\gamma_\xi|^2 e^{\varphi_\xi} - 1)|B^+|^2 - \frac{1-\lambda}{4}(|\gamma_\xi|^2 e^{\varphi_\xi} - 1)^2\right)\mathrm{d}x. \end{aligned} \tag{3.9}$$

Consider minimization in $B^\pm$ of the first line in the right hand side of (3.9). This yields the following Euler-Lagrange equations

$$\nabla^\perp h^+ = B^+ \text{ in } G \;\text{ and } \nabla^\perp h^- = 0 \text{ in } \omega, \tag{3.10}$$

and the boundary condition

$$h^+ = 0 \text{ on } \partial\Omega,$$

where $h^\pm = \mathrm{curl}B^\pm$. Since $A^{(\xi)} \in H^1(\Omega;\mathbb{R}^2)$ we also have the conjugation condition

$$B^+ + E^{(\xi)} = B^- \text{ on } \partial\omega. \tag{3.11}$$

The second equation in (3.10) implies that $h^- = \text{const}$, then in view of (3.11) we obtain

$$|\omega|h^- = \int_\omega h^-\mathrm{d}x = \int_{\partial\omega} B^-\cdot\tau\,\mathrm{d}s = \int_{\partial\omega}(B^+ + E^{(\xi)})\cdot\tau\,\mathrm{d}s, \tag{3.12}$$

that is

$$h^- = \frac{1}{|\omega|}\int_{\partial\omega}(B^+ + E^{(\xi)})\cdot\tau\,\mathrm{d}s.$$

Since for the actual critical points of (2.1) curl$A$ is continuous across $\partial\omega$, we require that $h^+ = h^-$ on $\partial\omega$. Then taking curl in the first equation in (3.10) we arrive at the following boundary value problem

$$\begin{cases} \Delta h^+ = h^+ \text{ in } G \\ h^+ = 0 \text{ on } \partial\Omega \\ h^+ = \dfrac{1}{|\omega|}\displaystyle\int_{\partial\omega}(B^+ + E^{(\xi)})\cdot\tau\,\mathrm{d}s \text{ on } \partial\omega. \end{cases}$$

According to (3.10) we have $B^+\cdot\tau = \partial h^+/\partial\nu$ on $\partial\omega$. This yields

$$h^+(x) = \Big(\frac{1}{K_G}\int_{\partial\omega}E^{(\xi)}\cdot\tau\,\mathrm{d}s\Big)V(x),$$

where

$$K_G = |\omega| + \int_G(|\nabla V|^2 + V^2)\,\mathrm{d}x,$$

and V is the unique solution of problem (1.4). We now define $B^\pm$ by $B^+ = \nabla^\perp h^+$ and $B^- = \nabla\chi + \nabla^\perp\mu$, where μ is a solution of

$$\begin{cases} \Delta\mu = h^- \text{ in } \omega \\ \frac{\partial\mu}{\partial\nu} = (B^+ + E^{(\xi)})\cdot\tau \text{ on } \partial\omega, \end{cases} \tag{3.13}$$

and $\chi \in H^2(\omega)$ is a function satisfying the boundary conditions $\chi = 0$ and $\frac{\partial\chi}{\partial\nu} = \frac{\partial\mu}{\partial\tau}$ on $\partial\omega$ (for the sake of definiteness we may assume that χ solves $\Delta^2\chi = 0$ in ω). Existence of a solution $\mu \in H^2(\omega)$ of problem (3.13) follows from (3.12). Then we have $B^-\cdot\tau = \frac{\partial\chi}{\partial\tau} + \frac{\partial\mu}{\partial\nu} = (B^+ + E^{(\xi)})\cdot\tau$ and $B^-\cdot\nu = \frac{\partial\chi}{\partial\nu} - \frac{\partial\mu}{\partial\tau} = 0$ on $\partial\omega$, while $(B^+ + E^{(\xi)})\cdot\nu = \frac{1}{2}\frac{\partial\varphi_\xi}{\partial\tau} - \frac{\partial h^+}{\partial\tau} = 0$ on $\partial\omega$ (since $\varphi_\xi, h^+ = \text{const}$ on $\partial\omega$). Thus $A^{(\xi)}$ defined by (3.1) belongs to $H^1(\Omega;\mathbb{R}^2)$.

We have constructed $(u^{(\xi)}, A^{(\xi)})$ which is an admissible testing pair, up to a gauge transformation, for the minimization problem (2.3). A straightforward calculation of $F_\lambda[u^{(\xi)}, A^{(\xi)}]$, that takes into account (3.9), yields the following upper bound,

$$\begin{aligned} m(\lambda) \le F_\lambda[u^{(\xi)}, A^{(\xi)}] &= \pi + \frac{1}{2K_G}\Big(\int_{\partial\omega}A^{(\xi)}\cdot\tau\,\mathrm{d}s\Big)^2 \\ &\quad + \frac{1}{2}\int_G\big((|\gamma_\xi|^2e^{\varphi_\xi} - 1)|B^+|^2 - \frac{1-\lambda}{4}(|\gamma_\xi|^2e^{\varphi_\xi} - 1)^2\mathrm{d}x \\ &\le \pi + \frac{1}{8K_G}\Big(\int_{\partial\omega}\frac{\partial\varphi_\xi}{\partial\nu}\,\mathrm{d}s\Big)^2 - \frac{1-\lambda}{8}\int_G(|\gamma_\xi|^2e^{\varphi_\xi} - 1)^2\mathrm{d}x, \end{aligned} \tag{3.14}$$

where we have also used the pointwise inequality $|\gamma_\xi|^2 e^{\varphi_\xi} \le 1$ in G which can be obtained by applying the maximum principle to the problem (3.3)-(3.4) (see Remark 4 in Section 8). The asymptotic behavior of the right hand side $I(\xi, \lambda)$ of (3.14) as $\xi \to \partial\Omega$ will be studied in Section 8. Namely, it will be shown that, if ξ^* denotes the nearest point projection of ξ on $\partial\Omega$ and $\delta = |\xi^* - \xi|$ is small, then

$$I(\xi,\lambda) = \pi + \frac{2\pi^2}{K_G}\delta^2\Big|\frac{\partial V}{\partial \nu}(\xi^*)\Big|^2 - \pi(1-\lambda)\delta^2|\log\delta| + o(\delta^2 + \delta^2|(1-\lambda)\log\delta|). \tag{3.15}$$

Letting $\delta \to +0$ in (3.15)(i.e. $\xi \to \partial\Omega$), we get

$$m(\lambda) \le \pi, \text{ for every } \lambda > 0, \tag{3.16}$$

and

$$m(\lambda) < \pi, \text{ for every } 0 < \lambda < 1. \tag{3.17}$$

4. Existence/nonexistence of minimizers

Bounds (3.16)-(3.17) allow us to resolve the question of attainability of the infimum $m(\lambda)$ in (2.3). We make use of the following result, which is a straightforward adaptation of Lemma 1 from [3].

Lemma 1. *Let $(u^{(n)}, A^{(n)}) \in \mathcal{J}_{01} \times H^1(\Omega;\mathbb{R}^2)$ be a sequence such that $(u^{(n)}, A^{(n)}) \to (u, A)$ weakly in $H^1(G;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$, then $u \in \mathcal{J}$ and*

$$\liminf_{n\to\infty} F_\lambda[u^{(n)}, A^{(n)}] \ge F_\lambda[u, A] + \pi(|\deg(u,\partial\Omega) - 1| + |\deg(u,\partial\omega)|).$$

Theorem 2. *(i) The infimum $m(\lambda)$ is always attained for $0 < \lambda < 1$, (ii) $m(\lambda)$ is never attained for $\lambda \ge 1$.*

Proof. (i) follows easily from (3.17) and Lemma 1. Indeed, let $(u^{(n)}, A^{(n)})$ be a minimizing sequence. By (3.17) this sequence is bounded in $H^1(G;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$. Thus, up to extracting a subsequence, $(u^{(n)}, A^{(n)}) \to (u, A)$ weakly in $H^1(G;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$. We need only to show that $u \in \mathcal{J}_{01}$. To this end, applying Lemma 1 we get

$$m(\lambda) = \liminf_{n\to\infty} F_\lambda[u^{(n)}, A^{(n)}] \ge F_\lambda[u, A] + \pi|1 - \deg(u,\partial\Omega)| + \pi|\deg(u,\partial\omega)|.$$

Since $m(\lambda) < \pi$, it follows that $\deg(u,\partial\Omega) = 1$ and $\deg(u,\partial\omega) = 0$, i.e. $u \in \mathcal{J}_{01}$.

Let us now show (ii). Assume by contradiction that (u, A) is a minimizer. By (2.8) and (3.16),

$$m(\lambda) = \pi + F^+[u, A] + \frac{1}{2}\int_\omega (\mathrm{curl} A)^2 \,\mathrm{d}x + \frac{\lambda - 1}{8}\int_G (|u|^2 - 1)^2 \,\mathrm{d}x \leq \pi.$$

Since $\lambda \geq 1$, we have

$$\frac{\partial u}{\partial \bar{z}} = \frac{iA_1 - A_2}{2} u \text{ in } G, \text{ and } \mathrm{curlA} = 0 \text{ in } \omega. \tag{4.1}$$

The first equation in (4.1) yields the following relation

$$\frac{\partial |u|^2}{\partial \nu} - 2u \wedge \frac{\partial u}{\partial \tau} + 2A \cdot \tau = 0 \text{ on } \partial\omega,$$

therefore, according to the second equation in (4.1) and the fact that $\deg(u, \partial\omega) = 0$,

$$\int_{\partial\omega} \frac{\partial |u|^2}{\partial \nu} \,\mathrm{d}s = 4\pi \deg(u, \partial\omega) - 2\int_\omega \mathrm{curlA} = 0.$$

On the other hand, by (2.7), $(|u|^2 - 1)/2$ solves

$$\begin{cases} \Delta \frac{|u|^2 - 1}{2} - \lambda |u|^2 \frac{|u|^2 - 1}{2} = |\nabla u - iAu|^2 \text{ in } G \\ \frac{|u|^2 - 1}{2} = 0 \text{ on } \partial G. \end{cases}$$

By the (strong) maximum principle and Hopf's lemma we have, either $|u| \equiv 1$ in G, or $|u| < 1$ in G and $\frac{\partial |u|^2}{\partial \nu} < 0$ on $\partial\omega$. It follows that $|u| \equiv 1$ in G and therefore $u \notin \mathcal{J}_{01}$. □

5. Lower bound

The upper bound construction of Section 3 provides the existence of minimizers (u^λ, A^λ) of problem (2.3) for every $0 < \lambda < 1$. In this section we show the optimality of this construction for $\lambda \to 1 - 0$. Namely, we prove

Lemma 2. *There exists a point ξ^λ such that $\xi^\lambda \to \partial\Omega$ as $\lambda \to 1 - 0$ and*

$$m(\lambda) = F_\lambda[u^\lambda, A^\lambda] \geq \pi + \frac{1}{8K_G}\Big(\int_{\partial\omega} \frac{\partial \varphi_{\xi^\lambda}}{\partial \nu} \,\mathrm{d}s\Big)^2 - (1 + o(1))\frac{1 - \lambda}{8}\int_G (|\gamma_{\xi^\lambda}|^2 e^{\varphi_{\xi^\lambda}} - 1)^2 \mathrm{d}x, \tag{5.1}$$

where γ_{ξ^λ}, φ_{ξ^λ} are defined by (3.5) and (3.3)-(3.4) with $\xi = \xi^\lambda$.

Proof. To get the result we study, in several steps, the asymptotic behavior of minimizers (u^λ, A^λ) as $\lambda \to 1-0$. As the **first step** we show that

$$\exists\, \Psi^\lambda = \text{const} \in \mathbb{S}^1 \text{ such that } u^\lambda - \Psi^\lambda \to 0 \text{ weakly in } H^1(G;\mathbb{C}), \tag{5.2}$$

$$A^\lambda \to 0 \text{ strongly in } H^1(\Omega;\mathbb{R}^2). \tag{5.3}$$

By the Sobolev embedding (5.2) will imply that

$$\int_G (|u^\lambda|^2 - 1)^2 \,\mathrm{d}x \to 0. \tag{5.4}$$

Thus, we can introduce a small positive parameter

$$\varepsilon(=\varepsilon(\lambda)) := \Big(\frac{1-\lambda}{8}\int_G (|u^\lambda|^2-1)^2\,\mathrm{d}x\Big)^{1/2}, \tag{5.5}$$

such that

$$\varepsilon^2/(1-\lambda) \to 0,$$

and write $F_\lambda[u^\lambda, A^\lambda]$ as (cf. (2.8))

$$F_\lambda[u^\lambda, A^\lambda] = \pi + F^+[u^\lambda, A^\lambda] + \frac{1}{2}\int_\omega |\mathrm{curl} A^\lambda|^2\,\mathrm{d}x - \varepsilon^2. \tag{5.6}$$

Proof of Claim (5.2)-(5.3). According to (3.17) we have $F_\lambda[u^\lambda, A^\lambda] = m(\lambda) < \pi$, therefore $\|u^\lambda\|_{H^1(G;\mathbb{C})} \le C$ and $\|A^\lambda\|_{H^1(\Omega;\mathbb{R}^2)} \le C$ with C independent of $0<\lambda<1$. Thus, up to extracting a subsequence, $(u^\lambda, A^\lambda) \to (u, A)$ weakly in $H^1(G;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$ as $\lambda \to 1-0$, where $u \in \mathcal{J}$. We have

$$F_1[u,A] \le \liminf_{\lambda\to 1-0} F_1[u^\lambda, A^\lambda] = \liminf_{\lambda \to 1-0} F_\lambda[u^\lambda, A^\lambda],$$

and, for every $v \in \mathcal{J}_{01}$ and $B \in H^1(\Omega;\mathbb{R}^2)$ satisfying (2.2),

$$F_\lambda[u^\lambda, A^\lambda] = m(\lambda) \le F_\lambda[v, B] \quad \forall\, 0<\lambda<1,$$

therefore $F_1[u, A] \le F_1[v, B]$. The infimum in (2.3) for $\lambda = 1$ is never attained, hence $u \notin \mathcal{J}_{01}$. Thus $|1 - \deg(u, \partial\Omega)| + |\deg(u, \partial\omega)| \ge 1$, and we have

$$\pi \ge \liminf_{\lambda\to 1-0} F_\lambda[u^\lambda, A^\lambda] = \liminf_{\lambda\to 1-0} F_1[u^\lambda, A^\lambda] \ge F_1[u, A] + \pi,$$

where we have used Lemma 1. We see that $F_1[u,A]=0$, hence $u=\text{const}\in\mathbb{S}^1$. This shows (5.2). To prove (5.3) we note that $\|A^\lambda\|_{H^1(\Omega;\mathbb{R}^2)}\leq C\|\text{curl}A^\lambda\|_{L^2(\Omega)}$ (with C independent of λ), thanks to the gauge choice (2.2). Then (3.16) and (5.8)-(5.6) imply that $\|A^\lambda\|_{H^1(\Omega;\mathbb{R}^2)}\to 0$ as $\lambda\to 1-0$. $\square$

Step II (*A priori bounds*). By (3.17), (5.6) and (2.9) we have

$$2\int_G\Big|\frac{\partial u^\lambda}{\partial\bar z}+\frac{A_2^\lambda-iA_1^\lambda}{2}u^\lambda\Big|^2\,\mathrm{d}x\leq\varepsilon^2, \tag{5.7}$$

$$\int_G(v^\lambda)^2\,\mathrm{d}x\leq 2\varepsilon^2,\quad |h_\omega^\lambda|^2\leq 2\varepsilon^2/|\omega|, \tag{5.8}$$

where

$$v^\lambda:=\text{curl}A^\lambda+\frac{1}{2}(|u^\lambda|^2-1),\ h_\omega^\lambda(=\text{const}):=\text{restriction of curl}A^\lambda\text{ to }\omega.$$

In Section 3 we have constructed testing pairs $(u^{(\xi)},A^{(\xi)})$ in a gauge such that $\text{div}A^{(\xi)}=0$ in G and $A^{(\xi)}\cdot\nu=0$ on ∂G. Now let us pass to such a gauge for minimizers (u^λ,A^λ) (A^λ was previously assumed to satisfy (2.2)). To this end consider a solution ψ^λ of the problem

$$\begin{cases}\Delta\psi^\lambda=0\text{ in }G\\ \frac{\partial\psi^\lambda}{\partial\nu}=0\text{ on }\partial\Omega,\ \frac{\partial\psi^\lambda}{\partial\nu}=-A^\lambda\cdot\nu\text{ on }\partial\omega.\end{cases}$$

Note that $A^\lambda+\nabla\psi^\lambda\to 0$ strongly in $L^2(G;\mathbb{R}^2)$ as $\lambda\to 1-0$, thanks to (5.3). Extend ψ^λ inside ω so that $\psi^\lambda\in H^2(\Omega)$, and perform the gauge change $u^\lambda\mapsto e^{i\psi^\lambda}u^\lambda$, $A^\lambda\mapsto A^\lambda+\nabla\psi^\lambda$. The new A^λ still belongs to $H^1(\Omega;\mathbb{R}^2)$ and

$$\text{div}A^\lambda=0\text{ in }G,\ A^\lambda\cdot\nu=0\text{ on }\partial\Omega\text{ and }\partial\omega. \tag{5.9}$$

Additionally, we have

$$\|A^\lambda\|_{L^2(G;\mathbb{R}^2)}\to 0\text{ as }\lambda\to 1-0. \tag{5.10}$$

Step III (*Asymptotic behavior of $v^\lambda=\text{curl}A^\lambda+\frac{1}{2}(|u^\lambda|^2-1)$*). Note that the Euler-Lagrange equation (2.5) implies that

$$\frac{\partial v^\lambda}{\partial\bar z}=\overline{u^\lambda}\Big(\frac{\partial u^\lambda}{\partial\bar z}+\frac{A_2^\lambda-iA_1^\lambda}{2}u^\lambda\Big)\text{ in }G. \tag{5.11}$$

By taking $\frac{\partial}{\partial z}$ of (5.11), on account of equation (2.4), we get

$$\begin{cases} \Delta v^\lambda - |u^\lambda|^2 v^\lambda = 4\Big|\dfrac{\partial u^\lambda}{\partial \bar z} + \dfrac{A_2^\lambda - iA_1^\lambda}{2}u^\lambda\Big|^2 + \dfrac{1-\lambda}{2}(1-|u^\lambda|^2)|u^\lambda|^2 \text{ in } G \\ v^\lambda = 0 \text{ on } \partial\Omega \\ v^\lambda = h_\omega^\lambda \text{ on } \partial\omega. \end{cases} \tag{5.12}$$

Set

$$\tilde v^\lambda := h_\omega^\lambda V \tag{5.13}$$

where V is the solution of problem (1.4), then

$$\Delta(v^\lambda - \tilde v^\lambda) - (v^\lambda - \tilde v^\lambda) = 4\Big|\frac{\partial u^\lambda}{\partial \bar z} + \frac{A_2^\lambda - iA_1^\lambda}{2}u^\lambda\Big|^2 + \frac{1-\lambda}{2}|u^\lambda|^2(1-|u^\lambda|^2) - (1-|u^\lambda|^2)v^\lambda \text{ in } G,$$

and $v^\lambda - \tilde v^\lambda = 0$ on ∂G. Owing to (5.7), the first bound in (5.8) and the pointwise inequality $|u^\lambda| \le 1$ in G, we can estimate the L^1-norm of the terms in the right hand of the equation as $2\varepsilon^2$, $(2(1-\lambda)|G|)^{1/2}\varepsilon$ and $4\varepsilon^2/(1-\lambda)^{1/2}(= o(\varepsilon))$, respectively. Therefore, by using well known estimates for elliptic equations with right hand side in L^1 (see, e.g., [13]), we find, as $\lambda \to 1-0$

$$\frac{1}{\varepsilon}\|v^\lambda - \tilde v^\lambda\|_{W^{1,p}(G)} \to 0 \text{ for every } 1 \le p < 2. \tag{5.14}$$

Step IV (*Change of unknowns*). We represent A^λ and u^λ as

$$A^\lambda = h_\omega^\lambda \nabla^\perp V + \tilde E^\lambda = h_\omega^\lambda \nabla^\perp V - \frac{1}{2}\nabla^\perp \tilde\varphi^\lambda \text{ and } u^\lambda = e^{\tilde\varphi^\lambda/2}(\theta^\lambda + w^\lambda),$$

where h_ω^λ is the restriction of $\mathrm{curl} A^\lambda$ to ω; V is the solution of problem (1.4); $\tilde\varphi^\lambda$ is a function which takes constant values on the connected components of ∂G and satisfies a certain partial differential equation (see problem (5.19) below); θ^λ satisfies $\Delta\theta^\lambda = 0$ in G; w^λ vanishes on ∂G and has a negligibly small H^2-norm (of order $o(\varepsilon)$). We will also get a lower bound for $F_\lambda[u^\lambda, A^\lambda]$ in terms of h_ω^λ, $\tilde\varphi^\lambda$, θ^λ and w^λ.

We begin the aforementioned transformations by setting

$$\tilde E^\lambda := A^\lambda - \nabla^\perp \tilde v^\lambda = A^\lambda - h_\omega^\lambda \nabla^\perp V. \tag{5.15}$$

Then, using (5.11) we obtain

$$\tilde{F}_\lambda[u^\lambda, \tilde{E}^\lambda] := F_\lambda[u^\lambda, A^\lambda] = \pi + F^+[u^\lambda, \tilde{E}^\lambda] - \frac{1-\lambda}{8}\int_G (|u^\lambda|^2 - 1)^2 \,\mathrm{d}x$$
$$+ 4\int_G \Big(\frac{\partial v^\lambda}{\partial \bar{z}} - \frac{\partial \tilde{v}^\lambda}{\partial \bar{z}}, \frac{\partial \tilde{v}^\lambda}{\partial \bar{z}}\Big)\,\mathrm{d}x + \int_G (v^\lambda - \tilde{v}^\lambda)\tilde{v}^\lambda \,\mathrm{d}x$$
$$+ \frac{1}{2}\int_G \big(|\nabla \tilde{v}^\lambda|^2 + (\tilde{v}^\lambda)^2\big)\,\mathrm{d}x + \frac{1}{2}\int_\omega |h^\lambda_\omega|^2 \,\mathrm{d}x$$
$$+ \frac{1}{2}\int_G |\nabla \tilde{v}^\lambda|^2 (1 - |u^\lambda|^2)\,\mathrm{d}x. \quad (5.16)$$

Due to the facts that $\Delta \tilde{v}^\lambda = \tilde{v}^\lambda$ in G and $v^\lambda = \tilde{v}^\lambda$ on ∂G, representation (5.16) is further simplified to

$$F_\lambda[u^\lambda, A^\lambda](= \tilde{F}_\lambda[u^\lambda, \tilde{E}^\lambda]) = \pi + F^+[u^\lambda, \tilde{E}^\lambda] - \frac{1-\lambda}{8}\int_G (|u^\lambda|^2 - 1)^2 \,\mathrm{d}x$$
$$+ \frac{(h^\lambda_\omega)^2}{2}\Big(K_G + \int_G |\nabla V|^2 (1 - |u^\lambda|^2)\,\mathrm{d}x\Big). \quad (5.17)$$

Note that, in view of (5.9),(5.13) and (5.15), $\mathrm{div}\tilde{E}^\lambda = 0$ in G and $\tilde{E}^\lambda \cdot \nu = 0$ on ∂G. Therefore there exists a potential $\tilde{\varphi}^\lambda$ such that

$$\tilde{E}^\lambda = -\frac{1}{2}\nabla^\perp \tilde{\varphi}^\lambda, \quad (5.18)$$

and $\tilde{\varphi}^\lambda$ takes constant values on $\partial\Omega$ and $\partial\omega$. Due to the fact that φ^λ is defined up to an additive constant, we can assume that the constant value of φ^λ on $\partial\Omega$ is zero. Then $\tilde{\varphi}^\lambda$ is the solution of the boundary value problem

$$\begin{cases} -\Delta \tilde{\varphi}^\lambda = 2\mathrm{curl}\tilde{E}^\lambda = 2(v^\lambda - \tilde{v}^\lambda) - |u^\lambda|^2 + 1 \text{ in } G \\ \tilde{\varphi}^\lambda = 0 \text{ on } \partial\Omega \\ \tilde{\varphi}^\lambda = \alpha^\lambda \text{ on } \partial\omega, \end{cases} \quad (5.19)$$

where α^λ is some constant. Since $|\nabla \tilde{\varphi}^\lambda| = 2|\tilde{E}^\lambda| \to 0$ strongly in $L^2(G)$ (by (5.10), (5.15) and the second bound in (5.8)), we know that $\alpha^\lambda \to 0$ as $\lambda \to 1-0$. We also know that for every $q \geq 1$ the L^q-norm of the right hand side in the above equation vanishes when $\lambda \to 1-0$, as follows from (5.4), (5.14) and the pointwise inequality $|u^\lambda| \le 1$ in G. Then by elliptic estimates we have

$$\tilde{\varphi}^\lambda \to 0 \text{ in } W^{2,q}(G)\ (\forall q \ge 1) \text{ and, in particular, in } C^1(\bar{G}). \tag{5.20}$$

This fact plays an important role in the further analysis.

Now introduce

$$\tilde{\theta}^\lambda := e^{-\tilde{\varphi}^\lambda/2} u^\lambda \tag{5.21}$$

Observe that

$$\frac{\partial \tilde{\varphi}^\lambda}{\partial \bar{z}} = -\tilde{E}_2^\lambda + i\tilde{E}_1^\lambda \text{ and therefore } \frac{\partial u^\lambda}{\partial \bar{z}} + \frac{\tilde{E}_2^\lambda - i\tilde{E}_1^\lambda}{2} u^\lambda = e^{\tilde{\varphi}^\lambda/2} \frac{\partial \tilde{\theta}^\lambda}{\partial \bar{z}}.$$

Since u^λ minimizes (5.17) with respect to its own boundary data, $\tilde{\theta}^\lambda$ satisfies the following equation

$$4\frac{\partial}{\partial z}\big(e^{\tilde{\varphi}^\lambda}\frac{\partial}{\partial \bar{z}}\tilde{\theta}^\lambda\big) = \big(\mathrm{curl}\tilde{E}^\lambda + \frac{\lambda}{2}(|u^\lambda|^2 - 1) - (h_\omega^\lambda)^2|\nabla V|^2\big)e^{\tilde{\varphi}^\lambda}\tilde{\theta}^\lambda \text{ in } G.$$

Next we pass from $\tilde{\theta}^\lambda$ to θ^λ, which satisfies $\Delta\theta^\lambda = 0$ in G, by setting

$$\theta^\lambda := \tilde{\theta}^\lambda - w^\lambda, \tag{5.22}$$

where w^λ is the unique solution of the equation

$$\Delta w^\lambda = -4\frac{\partial \tilde{\varphi}^\lambda}{\partial z}\frac{\partial \tilde{\theta}^\lambda}{\partial \bar{z}} + \big(\mathrm{curl}\tilde{E}^\lambda + \frac{\lambda}{2}(|u^\lambda|^2 - 1) - (h_\omega^\lambda)^2|\nabla V|^2\big)\tilde{\theta}^\lambda \text{ in } G, \tag{5.23}$$

subject to the boundary condition

$$w^\lambda = 0 \text{ on } \partial G. \tag{5.24}$$

By the very definition of θ^λ we have the following properties,

$$\Delta\theta^\lambda = 0 \text{ in } G; \tag{5.25}$$

$$|\theta^\lambda| = 1 \text{ on } \partial\Omega \text{ and } \deg(\theta^\lambda, \partial\Omega) = 1; \tag{5.26}$$

$$|\theta^\lambda| = \exp(-\alpha^\lambda/2) \text{ on } \partial\omega,\ \deg(\theta^\lambda/|\theta^\lambda|, \partial\omega) = 0$$
$$\text{(note that } \exp(-\alpha^\lambda/2) \to 1 \text{ as } \lambda \to 1-0, \text{ according to (5.20));} \tag{5.27}$$

Let us show that θ^λ also satisfy

$$\int_G \Big|\frac{\partial \theta^\lambda}{\partial \bar z}\Big|^2 \mathrm{d}x \le C\varepsilon^2. \tag{5.28}$$

Indeed, we observe that

$$\begin{aligned}\frac{\partial \theta^\lambda}{\partial \bar z} + \frac{\partial w^\lambda}{\partial \bar z} &= e^{-\tilde\varphi^\lambda/2}\Big(\frac{\partial u^\lambda}{\partial \bar z} + \frac{\tilde E_2^\lambda - i\tilde E_1^\lambda}{2} u^\lambda\Big)\\ &= e^{-\tilde\varphi^\lambda/2}\Big(\frac{\partial u^\lambda}{\partial \bar z} + \frac{A_2^\lambda - iA_1^\lambda}{2} u^\lambda - h_\omega^\lambda \frac{\partial V}{\partial \bar z} u^\lambda\Big). \end{aligned} \tag{5.29}$$

Then (5.28) immediately follows from (5.7)-(5.8), (5.20), the pointwise bound $|u^\lambda| \le 1$ in G and the following claim,

$$\frac{1}{\varepsilon}\|w^\lambda\|_{H^2(G)} \to 0 \text{ as } \lambda \to 1-0. \tag{5.30}$$

Proof of Claim (5.30). Since w^λ is a solution of problem (5.23)-(5.24), we have, by elliptic estimates,

$$\begin{aligned}\|w^\lambda\|_{H^2(G)} \le C\big(&\|\partial\tilde\varphi^\lambda/\partial z\|_{L^\infty(G;\mathbb{C})}\|\partial\tilde\theta^\lambda/\partial \bar z\|_{L^2(G;\mathbb{C})}\\ &+ \|\mathrm{curl}\tilde E^\lambda + \frac{1}{2}(|u^\lambda|^2-1)\|_{L^2(G)} + (1-\lambda)\||u^\lambda|^2-1\|_{L^2(G)} + (h_\omega^\lambda)^2\big),\end{aligned}$$

where we have also used the poinwise bound $|\tilde\theta^\lambda| = e^{-\tilde\varphi^\lambda/2}|u^\lambda| \le e^{-\tilde\varphi^\lambda/2} \le C$ in G (cf. (5.20)). Thanks to (5.7) and the second bound in (5.8), (5.15), (5.20) the following results hold,

$$\Big\|\frac{\partial\tilde\theta^\lambda}{\partial \bar z}\Big\|_{L^2(G;\mathbb{C})} \le \|e^{-\tilde\varphi^\lambda/2}\|_{L^\infty(G)}\Big\|\frac{\partial u^\lambda}{\partial \bar z} + \frac{\tilde E_2^\lambda - i\tilde E_1^\lambda}{2}u^\lambda\Big\|_{L^2(G;\mathbb{C})} = O(\varepsilon)$$

and $\|\partial\tilde\varphi^\lambda/\partial z\|_{L^\infty(G;\mathbb{C})} \to 0$ as $\lambda \to 1-0$, also $(1-\lambda)\||u^\lambda|^2-1\|_{L^2(G)} = o(\varepsilon)$ and $|h_\omega^\lambda| = O(\varepsilon)$. Besides $\mathrm{curl}\tilde E^\lambda + \frac{1}{2}(|u^\lambda|^2-1) = v^\lambda - \tilde v^\lambda$ while (5.14) implies that $\|v^\lambda - \tilde v^\lambda\|_{L^2(G)} = o(\varepsilon)$ (by the Sobolev embedding), and we are done. □

Finally, we note that, in view of the pointwise inequality $|u^\lambda| \le 1$ in G, (5.17) leads to the lower bound

$$F_\lambda[u^\lambda, A^\lambda] \ge \pi + \frac{K_G}{2}(h_\omega^\lambda)^2 - \frac{1-\lambda}{8}\int_G \big(e^{\tilde\varphi^\lambda}|\theta^\lambda + w^\lambda|^2 - 1\big)^2 \mathrm{d}x. \tag{5.31}$$

Step V (*Identification of* θ^λ, $\tilde\varphi^\lambda$). The following result is crucial.

Lemma 3. *The properties (5.25)-(5.28) of θ^λ imply that*

(i) θ^λ has exactly one zero ξ^λ when $\lambda \to 1-0$ and $\xi^\lambda \to \partial\Omega$; moreover, there are constants $C_1, C_2 > 0$ such that

$$C_1|\gamma_{\xi^\lambda}| \le |\theta^\lambda| \le C_2|\gamma_{\xi^\lambda}| \text{ in } G,$$

where γ_{ξ^λ} is defined by (3.5) with $\xi = \xi^\lambda$;

(ii) if, in addition,

$$\frac{1}{\varepsilon}\|\partial\theta^\lambda/\partial\bar{z}\|_{L^p(G;\mathbb{C})} \to 0 \text{ for some } p \ge 1, \tag{5.32}$$

then

$$\log|\theta^\lambda| - \log|\gamma_{\xi^\lambda}| = o(\varepsilon) \text{ on } \partial\omega$$

and there is ϑ^λ such that $|\vartheta^\lambda| \le C\varepsilon|\gamma_{\xi^\lambda}|$ in G,

$$\|\,|\vartheta^\lambda|\,\|_{L^q(G)} = o(\varepsilon), \quad \|\log(|\theta^\lambda - \vartheta^\lambda|/|\gamma_{\xi^\lambda}|)\|_{L^q(G)} = o(\varepsilon), \ \forall q \ge 1.$$

Remark 2. *Maps γ_{ξ^λ} in Lemma 3 can be regarded as projections of θ^λ on the (rigid) family of holomorphic maps defined by (3.5). Note that the constant value of $|\gamma_\xi|$ is uniquely determined by the zero ξ of γ_ξ. Thus, Lemma 3 allows, in particular, to reconstruct the unknown constant value of $|\theta^\lambda|$ on $\partial\omega$ via the unique zero ξ^λ of θ^λ (up to a negligibly small error). Additionally, it follows from Lemma 3 that $\|\,|\theta^\lambda|^2 - |\gamma_{\xi^\lambda}|^2\,\|_{L^q(G)} = o(\varepsilon)$ for every $q \ge 1$.*

The proof of Lemma 3 is presented in Section 6. Let us show that θ^λ satisfies condition (5.32) of Lemma 3. We note that, by (5.11) and (5.29),

$$\Big|\frac{\partial\theta^\lambda}{\partial\bar{z}} + \frac{\partial w^\lambda}{\partial\bar{z}}\Big| = \frac{e^{-\tilde{\varphi}^\lambda/2}}{|u^\lambda|}\Big|\frac{\partial v^\lambda}{\partial\bar{z}} - |u^\lambda|^2\frac{\partial\tilde{v}^\lambda}{\partial\bar{z}}\Big| \text{ when } |u^\lambda| > 0.$$

Due to (5.28), (5.30) we also have $\big\|\frac{1}{\varepsilon}\big|\frac{\partial\theta^\lambda}{\partial\bar{z}} + \frac{\partial w^\lambda}{\partial\bar{z}}\big|\,\big\|_{L^2(G)} \le C$. On the other hand,

$$\frac{1}{\varepsilon}\Big|\frac{\partial v^\lambda}{\partial\bar{z}} - |u^\lambda|^2\frac{\partial\tilde{v}^\lambda}{\partial\bar{z}}\Big| \le \frac{1}{\varepsilon}\Big|\frac{\partial v^\lambda}{\partial\bar{z}} - \frac{\partial\tilde{v}^\lambda}{\partial\bar{z}}\Big| + (1-|u^\lambda|^2)\frac{h_\omega^\lambda}{\varepsilon}\Big|\frac{\partial\, V}{\partial\bar{z}}\Big|$$

and the right hand side converges to zero in measure, as follows from (5.4), (5.14) and the second bound in (5.8). Then, using (5.4) and (5.20), we see that $\frac{1}{\varepsilon}|\frac{\partial\theta^\lambda}{\partial\bar z}+\frac{\partial w^\lambda}{\partial\bar z}|$ tends to zero in measure as $\lambda\to 1-0$. Therefore $\big\|\frac{1}{\varepsilon}|\frac{\partial\theta^\lambda}{\partial\bar z}+\frac{\partial w^\lambda}{\partial\bar z}|\big\|_{L^p(G)}\to 0$ for every $1\le p<2$. Finally, we make use of (5.30) to conclude that condition (5.32) is satisfied.

Using Lemma 3 we can identify the constant α^λ in problem (5.19),

$$\alpha^\lambda=-2\log|\gamma_{\xi^\lambda}(\partial\Omega)|+\kappa^\lambda \quad (|\gamma_{\xi^\lambda}|=\text{const on }\partial\Omega),$$

with the remainder κ^λ satisfying

$$\frac{1}{\varepsilon}|\kappa^\lambda|\to 0 \text{ as } \lambda\to 1-0. \tag{5.33}$$

Next, we identify $\tilde\varphi^\lambda$ by the following

Lemma 4. *Let ξ^λ be the unique zero of θ^λ (cf. Lemma 3), then*

$$\|\tilde\varphi^\lambda-\varphi_{\xi^\lambda}\|_{H^2(G)}=o(\varepsilon) \text{ as } \lambda\to 1-0,$$

where φ_{ξ^λ} is the solution of problem (3.3)-(3.4) with $\xi=\xi^\lambda$.

Proof. Set

$$f^\lambda:=\tilde\varphi^\lambda-\kappa^\lambda U,$$

where U is the unique solution of the equation $\Delta U=0$ in G subject to the boundary conditions $U=0$ on $\partial\Omega$ and $U=1$ on $\partial\omega$. Then f^λ satisfies $-\Delta f^\lambda=2(v^\lambda-\tilde v^\lambda)-e^{\tilde\varphi^\lambda}|\theta^\lambda+w^\lambda|^2+1$ in G (cf. (5.19), (5.21), (5.22)). Therefore, after simple calculations, we get the following boundary value problem for f^λ,

$$\begin{cases}-\Delta f^\lambda+|\gamma_{\xi^\lambda}|^2e^{f^\lambda}=1+r^\lambda \text{ in } G\\ f^\lambda=0 \text{ on } \partial\Omega\\ f^\lambda=-2\log|\gamma_{\xi^\lambda}| \text{ on } \partial\omega,\end{cases} \tag{5.34}$$

where

$$r^\lambda=2(v^\lambda-\tilde v^\lambda)+\big(|\gamma_{\xi^\lambda}|^2e^{-\kappa^\lambda U}-|\theta^\lambda-\vartheta^\lambda|^2\big)e^{\tilde\varphi^\lambda}\\ -\big(|\vartheta^\lambda+w^\lambda|^2+2(\theta^\lambda-\vartheta^\lambda,\vartheta^\lambda+w^\lambda)\big)e^{\tilde\varphi^\lambda}, \tag{5.35}$$

and ϑ^λ is as in Lemma 3. Let us show that L^2-norm of r^λ is negligibly small. To this end we use (5.14) and the Sobolev embedding for the first term of (5.35); for the last term we make use of statement (ii) of Lemma 3 and (5.30) in conjunction with the Sobolev embedding; finally, the middle term we represent as

$$\big((e^{-\kappa^\lambda U}-1)-(e^{2\log(|\theta^\lambda-\vartheta^\lambda|/|\gamma_{\xi^\lambda}|)}-1)\big)|\gamma_{\xi^\lambda}|^2e^{\tilde{\varphi}^\lambda}$$

and estimate it with the help of the elementary inequality $|e^t-1|\le|t|e^{|t|}$, Lemma 3, (5.20) and (5.33). As the result we get the following bound

$$\|r^\lambda\|_{L^2(G)}=o(\varepsilon). \tag{5.36}$$

This bound allows us to estimate the H^1-norm of the function $f^\lambda-\varphi_{\xi^\lambda}$. We have $-\Delta(f^\lambda-\varphi_{\xi^\lambda})+|\gamma_{\xi^\lambda}|^2e^{f^\lambda}-|\gamma_{\xi^\lambda}|^2e^{\varphi_{\xi^\lambda}}=r^\lambda$ in G. Multiply this equation by $f^\lambda-\varphi_{\xi^\lambda}$ to get, after integrating by parts,

$$\int_G|\nabla(f^\lambda-\varphi_{\xi^\lambda})|^2\mathrm{d}x\le\int_G|r^\lambda||f^\lambda-\varphi_{\xi^\lambda}|\mathrm{d}x$$

where we have used the monotonicity of the operator $\phi\mapsto|\gamma_{\xi^\lambda}|^2e^\phi$ and the fact that $f^\lambda-\varphi_{\xi^\lambda}=0$ on ∂G. It follows that

$$\|f^\lambda-\varphi_{\xi^\lambda}\|_{H^1(G)}=o(\varepsilon). \tag{5.37}$$

Next we show that the H^1-bound (5.37) in conjunction with an L^∞-estimate for f^λ (following from (5.20) and (5.33)) yield $\|\Delta(f^\lambda-\varphi_{\xi^\lambda})\|_{L^2(G)}=o(\varepsilon)$. By elliptic estimates this will imply that

$$\|f^\lambda-\varphi_{\xi^\lambda}\|_{H^2(G)}=o(\varepsilon). \tag{5.38}$$

In order to estimate $\Delta(f^\lambda-\varphi_{\xi^\lambda})$ we write

$$-\Delta(f^\lambda-\varphi_{\xi^\lambda})=r^\lambda-|\gamma_{\xi^\lambda}|^2\int_0^1(f^\lambda-\varphi_{\xi^\lambda})e^{(1-t)f^\lambda+t\varphi_{\xi^\lambda}}\,\mathrm{d}t,$$

to get, using the obvious pointwise pointwise inequality $|\gamma_{\xi^\lambda}|\le1$ in G,

$$\begin{aligned}\|\Delta(\varphi_{\xi^\lambda}+f^\lambda)\|_{L^2(G)}&\le\|r^\lambda\|_{L^2(G)}+\int_0^1\big\|\,|f^\lambda-\varphi_{\xi^\lambda}|e^{(1-t)f^\lambda+t\varphi_{\xi^\lambda}}\big\|_{L^2(G)}\mathrm{d}t\\&\le\|r^\lambda\|_{L^2(G)}+\|f^\lambda-\varphi_{\xi^\lambda}\|_{L^4(G)}e^{\|f^\lambda\|_{L^\infty(G)}}\int_0^1\|e^{2t(\varphi_{\xi^\lambda}-f^\lambda)}\|^{1/2}_{L^2(G)}\mathrm{d}t.\end{aligned}$$

Thus, in order to accomplish the proof of (5.38), it suffices to show that

$$\sup_{t\in[0,1]} \big\|\exp\big(2t(\varphi_{\xi^\lambda} - f^\lambda)\big)\big\|_{L^2(G)} \text{ remains bounded as } \lambda \to 1-0. \tag{5.39}$$

Indeed, according to (5.36) and (5.37) we have $\|r^\lambda\|_{L^2(G)}$, $\|f^\lambda - \varphi_{\xi^\lambda}\|_{L^4(G)} = o(\varepsilon)$, while $\|f^\lambda\|_{L^\infty(G)} \le \|\tilde\varphi^\lambda\|_{L^\infty(G)} + |\kappa^\lambda|\|U\|_{L^\infty(G)} \to 0$, as follows from (5.20) and (5.33).

It is straightforward to verify that for any $\phi \in H^1(G)$, $\phi \not\equiv 0$, and any $C_1 > 0$

$$\exp(2|\phi|) \le \exp(C_1\|\phi\|^2_{H^1}) \exp\Big(\frac{|\phi|^2}{C_1\|\phi\|^2_{H^1(G)}}\Big) \text{ in } G. \tag{5.40}$$

On the other hand, as shown in [14] (Chapter VII), there are $C_1, C_2 > 0$ such that

$$\int_G \exp\Big(\frac{|\phi|^2}{C_1\|\phi\|^2_{H^1(G)}}\Big)\mathrm{d}x \le C_2 \text{ for every } \phi \in H^1(G),\ \phi \not\equiv 0.$$

Therefore, integrating (5.40) over G, we get $\|\exp(\phi)\|_{L^2(G)} \le C\exp(C_1\|\phi\|^2_{H^1(G)})$. Then (5.37) implies that (5.39) does hold, and thus (5.38) is proved. Finally, since $\tilde\varphi^\lambda = f^\lambda + \kappa^\lambda U$ and $\kappa^\lambda = o(\varepsilon)$, the claim of the Lemma follows. □

Step IV (*Derivation of the lower bound*). Using (5.15), (5.18) and the definition of h^λ_ω (h^λ_ω is the restriction of $\mathrm{curl}A^\lambda$ to ω), we get

$$-\frac{1}{2}\int_{\partial\omega}\frac{\partial\tilde\varphi^\lambda}{\partial\nu}\mathrm{d}s = \int_\omega \mathrm{curl}A^\lambda\,\mathrm{d}x - h^\lambda_\omega\int_{\partial\omega}\frac{\partial V}{\partial\nu}\mathrm{d}s = h^\lambda_\omega\Big(|\omega| - \int_{\partial\omega}\frac{\partial V}{\partial\nu}\mathrm{d}s\Big) = h^\lambda_\omega K_G$$

Hence, by Lemma 4,

$$(h^\lambda_\omega)^2 = \frac{1}{4K_G^2}\Big(\int_{\partial\omega}\frac{\partial\varphi_{\xi^\lambda}}{\partial\nu}\mathrm{d}s\Big)^2 + o(\varepsilon^2). \tag{5.41}$$

It is not hard to show also that, by (5.5), (5.21)-(5.22), (5.30), Lemma 4 and (ii) of Lemma 3,

$$\varepsilon^2 = \frac{1-\lambda}{8}\int_G\big(e^{\tilde\varphi^\lambda}|\theta^\lambda + w^\lambda|^2 - 1\big)^2\mathrm{d}x = (1+o(1))\frac{1-\lambda}{8}\int_G\big(|\gamma_{\xi^\lambda}|^2 e^{\varphi_{\xi^\lambda}} - 1\big)^2\mathrm{d}x. \tag{5.42}$$

Now substitute (5.41) and (5.42) in (5.31) to get (5.1). Lemma 2 is proved. □

6. Proof of the key lemma

This section is devoted to the

Proof of Lemma 3. In the proof we will repeatedly make use of the formula

$$\int_G |\nabla u|^2 \,\mathrm{d}x = \frac{1}{2}\int_G \Big|\frac{\partial u}{\partial \bar z}\Big|^2 \mathrm{d}x \\ + \pi(|u(\partial\Omega)|^2 \deg(u/|u|, \partial\Omega) - |u(\partial\omega)|^2 \deg(u/|u|, \partial\omega)), \quad (6.1)$$

that is valid for any $u \in H^1(G;\mathbb{C})$ satisfying $|u| = \mathrm{const} > 0$ on $\partial\Omega$ and on $\partial\omega$ (with possibly another constant). To see (6.1) one integrates the pointwise identity

$$|\nabla u|^2 = 2\frac{\partial u}{\partial x_1} \wedge \frac{\partial u}{\partial x_2} + \frac{1}{2}\Big|\frac{\partial u}{\partial \bar z}\Big|^2 = \frac{\partial}{\partial x_1}\Big(u \wedge \frac{\partial u}{\partial x_2}\Big) - \frac{\partial}{\partial x_2}\Big(u \wedge \frac{\partial u}{\partial x_1}\Big) + \frac{1}{2}\Big|\frac{\partial u}{\partial \bar z}\Big|^2$$

over G and applies the divergence theorem.

We first show

Lemma 5. *We have*

$$|\theta^\lambda|^2 \to 1 \text{ strongly in } L^2(G) \text{ as } \lambda \to 1 - 0 \quad (6.2)$$

and

$$\nabla\tilde\theta^\lambda \to 0 \text{ in } C_{\mathrm{loc}}(G;\mathbb{C}^2). \quad (6.3)$$

Proof. Since θ^λ satisfies $\Delta\theta^\lambda = 0$ in G, we have

$$\Delta|\theta^\lambda|^2 = 2|\nabla\theta^\lambda|^2 \ge 0 \text{ in } G. \quad (6.4)$$

Then, by the maximum principle, $|\theta^\lambda| \le \max\{1, e^{-\alpha^\lambda/2}\}$ in G. Besides, by (5.28),

$$\frac{1}{2}\int_G |\nabla\theta^\lambda|^2 \,\mathrm{d}x = \pi + \frac{1}{4}\int_G \Big|\frac{\partial\theta^\lambda}{\partial\bar z}\Big|^2 \mathrm{d}x \le \pi + C\varepsilon^2, \quad (6.5)$$

where we have used formula (6.1). It follows that $\|\theta^\lambda\|_{H^1(G;\mathbb{C})} \le C$ with a constant C independent of λ. Therefore, up to extracting a subsequence, $\theta^\lambda \to \theta$ weakly in $H^1(G;\mathbb{C})$, and $|\theta| = 1$ on ∂G. Moreover, in view of (5.28) and (6.5),

$$\int_G \Big|\frac{\partial\theta}{\partial\bar z}\Big|^2 \mathrm{d}x = 0, \text{ i.e. } \frac{\partial\theta}{\partial\bar z} = 0 \text{ in } G,$$

and

$$\frac{1}{2}\int_G |\nabla\theta|^2\,\mathrm{d}x \le \pi. \tag{6.6}$$

It follows that $\theta = \mathrm{const} \in \mathbb{S}^1$. Indeed, since $\frac{\partial\theta}{\partial\bar{z}} = 0$ in G, it suffices to show that $|\theta| \equiv 1$ in G. We have $\Delta|\theta|^2 = 2|\nabla\theta|^2 + 2(\Delta\theta,\theta) = 2|\nabla\theta|^2 \ge 0$ in G and $|\theta| = 1$ on ∂G. If we assume that $|\theta| \not\equiv 1$ in ∂G, we obtain, by using Hopf's lemma, that $\frac{\partial|\theta|}{\partial\nu} > 0$ on $\partial\Omega$ and $\frac{\partial|\theta|}{\partial\nu} < 0$ on $\partial\omega$. The equation $\frac{\partial\theta}{\partial\bar{z}} = 0$ in G implies that $\theta \wedge \frac{\partial\theta}{\partial\tau} = \frac{\partial|\theta|}{\partial\nu} > 0$ on $\partial\Omega$ and $u \wedge \frac{\partial\theta}{\partial\tau} = \frac{\partial|\theta|}{\partial\nu} < 0$ on $\partial\omega$, consequently $\deg(\theta,\partial\Omega) \ge 1$ and $\deg(\theta,\partial\omega) \le -1$. Hence, by using formula (6.1), we get

$$\frac{1}{2}\int_G |\nabla\theta|^2\,\mathrm{d}x = \pi\deg(\theta,\partial\Omega) - \pi\deg(\theta,\partial\omega) \ge 2\pi,$$

and thus obtain a contradiction with (6.6). We have shown that, up to extracting a subsequence, $\theta^\lambda \to \mathrm{const} \in \mathbb{S}^1$ weakly in $H^1(G;\mathbb{C})$ as $\lambda \to 1-0$. The statement of the Lemma follows by the Sobolev embedding and elliptic estimates. □

We next study the pointwise asymptotic behavior of $|\theta^\lambda|$ to get the

Proof of (i) of Lemma 3. Since $\deg(\theta^\lambda,\partial\Omega) = 1$ and $\deg(\theta^\lambda/|\theta^\lambda|,\partial\omega) = 0$, θ^λ has at least one zero in G. Let ξ^λ be a zero of θ^λ nearest to $\partial\Omega$ then, by (6.2)-(6.3),

$$\xi^\lambda \to \partial\Omega \text{ as } \lambda \to 1-0.$$

Let us prove that ξ^λ is the unique zero. To this end we first show that other zeros (if exist) are localized near ξ^λ. We use the coarea formula of H. Federer and W.H. Fleming (see, e.g., [12]) to compute

$$\int_G |1-|\theta^\lambda|^2|\,|\nabla|\theta^\lambda||\,\mathrm{d}x = \int_0^{\max\{1,\exp(-\alpha^\lambda/2)\}} \mathrm{d}t \int_{\{x:|\theta^\lambda(x)|=t\}} |1-t^2|\;\mathrm{d}\mathcal{H}^1,$$

where $\mathcal{H}^1$ is 1-dimensional Hausdorff measure on $\mathbb{R}^2$. On the other hand, by the Cauchy-Schwarz inequality, (6.2) and (6.5), we obtain

$$\int_G |1-|\theta^\lambda|^2|\,|\nabla|\theta^\lambda||\,\mathrm{d}x \le C\|1-|\theta^\lambda|^2\|_{L^2(G)} \to 0 \text{ as } \lambda \to 1-0.$$

It follows that there is a regular value $t^\lambda \in (4/5, 6/7)$ of $|\theta^\lambda|$ such that $\mathcal{H}^1(\{x \in G; |\theta^\lambda| = t^\lambda\}) \to 0$, as $\lambda \to 1-0$. (Note that by Sard's lemma almost all $t \in (0, \max\{1, \exp(-\alpha^\lambda/2)\})$ are regular values of $|\theta^\lambda|$.) Set

$$T^\lambda := \{z \in G; |\theta^\lambda| < t^\lambda\},$$

then, assuming that $1-\lambda$ is sufficiently small, the boundary ∂T^λ of T^λ consists of a finite collection of $k(=k(\lambda))$ closed curves enclosing simply connected subdomains $\varpi_0^\lambda, \ldots, \varpi_k^\lambda$ of G, where ϖ_0^λ is a subdomain containing ξ^λ. By the (strong) maximum principle applied to (6.4) we have $|\theta^\lambda| < t^\lambda$ in each ϖ_j^λ. This means, in particular, that these domains are disjoint. Moreover, the following Lemma shows that for sufficiently small $1-\lambda$ we have

$$|\theta^\lambda| \geq 1/5 \text{ in } T^\lambda \setminus \varpi_0^\lambda. \tag{6.7}$$

Lemma 6. *Let ϖ be a simply connected domain with a smooth boundary and let $v \in H^1(\varpi, \mathbb{C})$ satisfy $\Delta v = 0$ in ϖ and $|v| \geq 4/5$ on $\partial\varpi$. Then, if $|v(y)| \leq 1/5$ at a point $y \in \varpi$, we have*

$$\frac{1}{2}\int_\varpi |\nabla v|^2 \, \mathrm{d}x \geq \frac{3\pi}{5}.$$

Proof. Since the equation $\Delta v = 0$ and the Dirichlet integral are conformally invariant, we can assume, without loss of generality, that $\varpi = B_1$ and $|v(0)| \leq 1/5$. Then

$$v = v(0) + \sum_{k=1}^\infty b_k z^k + c_k \bar{z}^k \text{ in } B_1(0),$$

and the Dirichlet integral is expressed as

$$\frac{1}{2}\int_\varpi |\nabla v|^2 \, \mathrm{d}x = \pi \sum_{k=1}^\infty k(|b_k|^2 + |c_k|^2),$$

while

$$\frac{16}{25}\pi \leq \frac{1}{2}\int_{\mathbb{S}^1} |v|^2 \, \mathrm{d}s = \pi(|v(0)|^2 + \sum_{k=1}^\infty (|b_k|^2 + |c_k|^2).$$

Therefore

$$\frac{1}{2}\int_\varpi |\nabla v|^2 \, \mathrm{d}x \geq \pi\Big(\frac{16}{25} - \frac{1}{25}\Big) = \frac{3\pi}{5}.$$

□

Proof of (i) of Lemma 3 completed. Lemma 6 in conjunction with (6.5) imply that zero ξ^λ lies in ϖ_0^λ, when λ is sufficiently close to 1. Besides, according to (6.7),

$$|\theta^\lambda| \geq \min\Big\{\inf_{T^\lambda\setminus\varpi_0^\lambda}|\theta^\lambda|, \inf_{G\setminus T^\lambda}|\theta^\lambda|\Big\} \geq 1/5 \text{ in } G\setminus\varpi_0^\lambda.$$

In order to study θ^λ in ϖ_0^λ we perform the rescaling by means of the conformal map a_{ξ^λ}, given by (3.6). Prior to that we extend θ^λ into ω in order to have $\|\theta^\lambda\|_{L^\infty(\Omega;\mathbb{C})}$, $\|\theta^\lambda\|_{H^1(\Omega;\mathbb{C})} \leq C$ with C independent of λ (it is possible because of L^∞- and H^1-bounds already established in the proof of Lemma 5). Set

$$\Theta^\lambda(\zeta) := \theta^\lambda(a_{\xi^\lambda}^{-1}(\zeta)).$$

Thanks to the conformal invariance of the Dirichlet integral we have $\Theta^\lambda \in H^1(B_1(0);\mathbb{C})$ and $\|\Theta^\lambda\|_{H^1(B_1(0);\mathbb{C})} \leq C$. Moreover, Θ^λ satisfies $\Delta\Theta^\lambda = 0$ in $a_{\xi^\lambda}(G)$ and $\Theta^\lambda(0) = \theta^\lambda(\xi^\lambda) = 0$. Without loss of generality we may assume that $\frac{\partial\Theta^\lambda}{\partial\zeta}(0)$ is real and $\frac{\partial\Theta^\lambda}{\partial\zeta}(0) \geq 0$ (this always can be achieved by multiplying θ^λ by a constant with modulus one). We claim that

$$\Theta^\lambda(\zeta) \to \zeta \text{ weakly in } H^1(B_1(0);\mathbb{C}) \text{ as } \lambda \to 1-0.$$

Clearly, up to extracting a subsequence, Θ^λ converges to some Θ weakly in $H^1(B_1(0);\mathbb{C})$ as $\lambda \to 1-0$, and $|\Theta| = 1$ on $\mathbb{S}^1 = \partial B_1(0)$. One easily checks that $|a_\xi(x)| \to 1$ uniformly on $\bar\omega$ as $\xi \to \partial\Omega$, therefore for any fixed $0 < r < 1$ we have $a_{\xi^\lambda}^{-1}(B_r(0)) \subset G$ when $1-\lambda$ is sufficiently small. For such λ, Θ^λ satisfies $\Delta\Theta^\lambda = 0$ in $B_r(0)$, consequently elliptic estimates imply the following convergence result,

$$\Theta^\lambda \to \Theta \text{ in } C^k(B_r(0);\mathbb{C}) \text{ for every } k > 0. \tag{6.8}$$

We have, in particular,

$$\Theta(0) = \lim_{\lambda\to1-0}\Theta^\lambda(0) = 0 \text{ and } \frac{\partial\Theta}{\partial\zeta}(0) = \lim_{\lambda\to1-0}\frac{\partial\Theta^\lambda}{\partial\zeta}(0) \geq 0.$$

Besides, using (6.5) we see that

$$\begin{aligned}\int_{B_1(0)}|\nabla\Theta|^2\,\mathrm{d}\zeta &= \lim_{r\to1-0}\lim_{\lambda\to1-0}\int_{B_r(0)}|\nabla\Theta^\lambda|^2\,\mathrm{d}\zeta\\ &= \lim_{r\to1-0}\lim_{\lambda\to1-0}\int_{a_{\xi^\lambda}^{-1}(B_r(0))}|\nabla\theta^\lambda|^2\,\mathrm{d}x \leq \lim_{\lambda\to1-0}\int_G|\nabla\theta^\lambda|^2\,\mathrm{d}x \leq 2\pi.\end{aligned} \tag{6.9}$$

On the other hand $\Delta\Theta = 0$ in $B_1(0)$, as follows from (6.8). Hence Θ can be represented as $\Theta = \sum_{k=1}^{\infty}(b_k\zeta^k + c_k\bar\zeta^k)$, and we can compute

$$\int_{B_1(0)} |\nabla\Theta|^2 \,\mathrm{d}\zeta - 2\pi = \int_{B_1(0)} |\nabla\Theta|^2 \,\mathrm{d}\zeta - \int_{\mathbb{S}^1} |\Theta|^2 \,\mathrm{d}s = 2\pi\sum_{k=1}^{\infty}(k-1)(|b_k|^2 + |c_k|^2).$$

Then (6.9) holds only if $b_k = c_k = 0$ for $k > 1$, i.e. $\Theta = b_1\zeta + c_1\bar\zeta$. Since $|b_1|^2 + |c_1|^2 = 1$, $b_1 = \frac{\partial\Theta}{\partial\zeta}(0) \geq 0$ and

$$|c_1|^2 = \frac{4}{\pi}\int_{B_{1/2}(0)} \Big|\frac{\partial\Theta}{\partial\bar\zeta}\Big|^2 \mathrm{d}\zeta = \lim_{\lambda\to 1-0}\frac{4}{\pi}\int_{a_{\xi^\lambda}^{-1}(B_{1/2}(0))} \Big|\frac{\partial\theta^\lambda}{\partial\bar z}\Big|^2 \mathrm{d}\zeta = 0 \text{ (by (5.28))},$$

we conclude that $\Theta(\zeta) = \zeta$.

Now from (6.8) we see that $\varpi_0^\lambda \subset a_{\xi^\lambda}^{-1}(B_{7/8}(0))$ when $1-\lambda$ is sufficiently small (since $|\theta^\lambda| = t^\lambda$ on $\partial\varpi_0^\lambda$ and $t^\lambda \in (4/5, 6/7)$ while

$$\min\{|\theta^\lambda(x)|; x \in \partial a_{\xi^\lambda}^{-1}(B_{7/8}(0))\} \to 7/8);$$

(6.8) also implies that $|\Theta^\lambda(\zeta)| = |\zeta|(1+o(1))$ in $B_{7/8}(0)$ as $\lambda \to 1-0$, or $|\theta^\lambda| = |a_{\xi^\lambda}|(1+o(1))$ in $a_{\xi^\lambda}^{-1}(B_{7/8}(0))$, where $o(1)$ stands for a function whose L^∞-norm vanishes in the limit. On the other hand, by (5.27) and (6.7), we have $\log(1/5) \leq \log|\theta^\lambda| \leq \max\{0, -\alpha^\lambda/2\} \leq C$ in $G \setminus \varpi_0^\lambda$. Thus ξ^λ is the unique zero of θ^λ. Moreover $C_1|a_{\xi^\lambda}| \leq |\theta^\lambda| \leq C_2|a_{\xi^\lambda}|$ in G for some constants $0 < C_1 < C_2$. In remains to note only that $|\gamma_{\xi^\lambda}|$ admits the factorization $|\gamma_{\xi^\lambda}| = |a_{\xi^\lambda}|\exp(\sigma_{\xi^\lambda})$ (see Section 3) and $\sigma_{\xi^\lambda} \to 0$ uniformly on $\bar G$ when $\xi^\lambda \to \partial\Omega$. □

Let us next introduce ϑ^λ satisfying the requirements in (ii) of Lemma 3.

Since the unique zero ξ^λ of θ^λ tends to $\partial\Omega$ as $\lambda \to 1-0$, we can assume that $a_{\xi^\lambda}^{-1}(B_{8/9}(0)) \subset G$. Rescaling θ^λ as above, $\Theta^\lambda(\zeta) = \theta^\lambda(a_{\xi^\lambda}^{-1}(\zeta))$, we have $\Delta\Theta_\lambda = 0$ in $B_{8/9}(0)$ and $\Theta^\lambda(0) = 0$. It follows that Θ^λ admits the representation

$$\Theta^\lambda(\zeta) = \sum_{k=1}^{\infty}(b_{k,\lambda}\zeta^k + c_{k,\lambda}\bar\zeta^k) \text{ in } B_{8/9}(0).$$

We set $\tilde\vartheta^\lambda$ to be the antiholomorphic part of Θ^λ,

$$\tilde\vartheta^\lambda := \sum_{k=1}^{\infty} c_{k,\lambda}\bar\zeta^k,$$

and show that

$$|\tilde{\vartheta}^\lambda(\zeta)| \le C\varepsilon|\zeta| \text{ in } B_{7/8}(0), \tag{6.10}$$

$$|\nabla\tilde{\vartheta}^\lambda| \le C\varepsilon \text{ in } B_{7/8}(0). \tag{6.11}$$

Both these bounds follow from the estimate $|c_{k,\lambda}| \le C(9/8)^k\varepsilon$, where C is independent of k and ε. The latter estimate is verified as follows,

$$\pi\sum_{k=1}^{\infty} k(8/9)^{2k}|c_{k,\lambda}|^2 = \int_{B_{8/9}(0)}\Big|\frac{\partial\Theta^\lambda}{\partial\bar{\zeta}}\Big|^2\,\mathrm{d}\zeta \le \int_G\Big|\frac{\partial\theta^\lambda}{\partial\bar{z}}\Big|^2\,\mathrm{d}x,$$

due to (5.28) the right hand side is bounded by $C\varepsilon^2$.

Now introduce ϑ^λ by

$$\vartheta^\lambda(z) := \sigma(a_{\xi^\lambda}(x))\tilde{\vartheta}^\lambda(a_{\xi^\lambda}(x)),$$

where σ is a smooth cut-off function such that

$$\sigma(\zeta) = \begin{cases} 1 \text{ if } \zeta \in B_{1/4}(0) \\ 0 \text{ if } \zeta \notin B_{1/2}(0). \end{cases}$$

Lemma 7. *We have*

$$|\vartheta^\lambda(z)| \le C\varepsilon|\gamma_{\xi^\lambda}(z)| \ \textit{in } G, \tag{6.12}$$

$$\int_G |\nabla\vartheta^\lambda(x)|^2\,\mathrm{d}x \le C\varepsilon^2 \ \textit{and} \ \int_G |\nabla\vartheta^\lambda(x)|^p\,\mathrm{d}x = o(\varepsilon^p) \ \textit{for every } 1 \le p < 2, \tag{6.13}$$

$$\frac{\partial}{\partial\bar{z}}(\theta^\lambda - \vartheta^l) = 0 \ \textit{in } a_{\xi^\lambda}^{-1}(B_{1/4}(0)). \tag{6.14}$$

Proof. Bound (6.12) follows from (6.10) and the pointwise inequality $|a_{\xi^\lambda}| \le |\gamma_{\xi^\lambda}|$ in G (this inequality can be easily derived from the constructive definition of $|\gamma_{\xi^\lambda}|$ given in Section 3); (6.14) is a straightforward consequence of the very definition of ϑ^λ. To show the first bound in (6.13) we argue by the conformal invariance of the Dirichlet integral,

$$\int_G |\nabla\vartheta^\lambda(x)|^2\,\mathrm{d}x = \int_{B_{1/2}(0)} |\nabla(\sigma(\zeta)\tilde{\vartheta}^\lambda(\zeta))|^2\,\mathrm{d}\zeta,$$

and make use of (6.10)-(6.11). Finally the second bound in (6.13) follows from the first one and the fact that the measure of $\operatorname{supp}(|\nabla\vartheta^\lambda|)$ tends to zero as $\lambda \to 1-0$. □

Note that the second bound in (6.13) in conjunction with the fact that $\vartheta^\lambda = 0$ on ∂G imply, by the Sobolev embedding, that $\|\vartheta^\lambda\|_{L^q(G)} = o(\varepsilon)$ for every $q \geq 1$. In order to complete the proof of (ii), we need to estimate $\log |s^\lambda|$, where

$$s^\lambda := (\theta^\lambda - \vartheta^\lambda)/\gamma_{\xi^\lambda}.$$

Observe that $0 < C_1 \leq |s^\lambda| \leq C_2$ when $1-\lambda$ is sufficiently small, which follows from (i) and (6.12). We also have $|s^\lambda| = 1$ on ∂G and $|s^\lambda| = \text{const} > 0$ on $\partial\omega$, moreover $\deg(s^\lambda, \partial\Omega) = \deg(s^\lambda/|s^\lambda|, \partial\omega) = 0$. Thus, we can fix a single-valued branch of $\log s^\lambda$ on G, and set

$$S^\lambda := \frac{1}{\varepsilon}\log s^\lambda$$

Lemma 8. *The real part of S^λ converges weakly in $H^1(G)$ to zero as $\lambda \to 1-0$.*

Proof. We have

$$\begin{aligned}\int_G |\nabla S^\lambda|^2 \,\mathrm{d}x &= \frac{1}{\varepsilon^2}\int_G |\nabla s^\lambda|^2 \frac{\mathrm{d}x}{|s^\lambda|^2} \leq \frac{C}{\varepsilon^2}\int_G |\nabla s^\lambda|^2 \,\mathrm{d}x \\ &= \frac{4C}{\varepsilon^2}\int_G \Big|\frac{\partial s^\lambda}{\partial \bar z}\Big|^2 \mathrm{d}x = \frac{4C}{\varepsilon^2}\int_{G\setminus a_{\xi^\lambda}^{-1}(B_{1/4}(0))} \Big|\frac{\partial s^\lambda}{\partial \bar z}\Big|^2 \mathrm{d}x \\ &\leq \frac{C_1}{\varepsilon^2}\int_G \Big(\Big|\frac{\partial \theta^\lambda}{\partial \bar z}\Big|^2 + \Big|\frac{\partial \vartheta^\lambda}{\partial \bar z}\Big|^2\Big)\,\mathrm{d}x \leq C_2,\end{aligned}$$

where we successively used the pointwise bound $1/|s^\lambda|^2 \leq C$, formula (6.1), property (6.14), the bound $|\gamma_{\xi^\lambda}| \geq |a_{\xi^\lambda}| \geq 1/4$ in $G \setminus a_{\xi^\lambda}^{-1}(B_{1/4}(0))$, and (5.28) together with the first bound in (6.13). The real part S_1^λ of S^λ satisfies $S_1^\lambda = 0$ on $\partial\Omega$, therefore, after subtracting the mean value $\langle S_2^\lambda\rangle$ from the imaginary part, we get

$$\|S^\lambda - i\langle S_2^\lambda\rangle\|_{H^1(G;\mathbb{C})} \leq C.$$

Thus, up to extracting a subsequence,

$$S^\lambda - i\langle S_2^\lambda\rangle \to S \text{ weakly in } H^1(G;\mathbb{C}),$$

where $S \in H^1(G;\mathbb{C})$ and its real part vanishes on $\partial\Omega$ and takes constant values on $\partial\omega$ while the imaginary part has zero mean over G. On the other hand

$$\int_G\Big|\frac{\partial S^\lambda}{\partial\bar z}\Big|^p\,\mathrm{d}x \le \frac{C}{\varepsilon^p}\int_{G\setminus a_{\xi^\lambda}^{-1}(B_{1/4}(0))}\Big(\Big|\frac{\partial\theta^\lambda}{\partial\bar z}\Big|^p+\Big|\frac{\partial\vartheta^\lambda}{\partial\bar z}\Big|^p\Big)\frac{\mathrm{d}x}{|\gamma_{\xi^\lambda}|^p}$$
$$\le \frac{C}{\varepsilon^p}\int_G\Big(\Big|\frac{\partial\theta^\lambda}{\partial\bar z}\Big|^p+\Big|\frac{\partial\vartheta^\lambda}{\partial\bar z}\Big|^p\Big)\,\mathrm{d}x,$$

and according to (5.32) and the second bound in (6.13) the right hand side tends to zero as $\lambda\to 1-0$. Thus

$$\frac{\partial S}{\partial\bar z}=0 \text{ in } G. \tag{6.15}$$

It follows that $\exp(S):G\to\mathbb{C}$ is a holomorphic map, $|\exp(S)|=1$ on $\partial\Omega$, $|\exp(S)|=\text{const}$ on $\partial\omega$ and $\deg(\exp(S),\partial\Omega)=\deg(\exp(S)/|\exp(S)|,\partial\omega)=0$ (since the imaginary part of S is a single valued function). Thus, by (6.15),

$$\frac{1}{2}\int_G|\nabla\exp(S)|^2\,\mathrm{d}x=2\int_G\Big|\frac{\partial\exp(S)}{\partial\bar z}\Big|^2\,\mathrm{d}x=0, \tag{6.16}$$

where we have used formula (6.1). Hence $S\equiv 0$ in G, because (6.16) implies that S is a constant while the real part of S vanishes on $\partial\Omega$ and its imaginary part has zero mean. □

Lemma 8 implies the convergence of traces, $|S^\lambda|\to 0$ on $\partial\omega$, i.e. $\log|\theta^\lambda|-\log|\gamma_{\xi^\lambda}|=o(\varepsilon)$, and also, by the Sobolev embedding, the convergence of $S^\lambda=\frac{1}{\varepsilon}\log(|\theta^\lambda-\vartheta^\lambda|/|\gamma_{\xi^\lambda}|)$ in $L^q(G)$ to zero for every $q\ge 1$. Lemma 3 is proved. □

7. Near boundary vortices and δ-like behavior of currents

In this section we analyze the behavior of vortices of minimizers as $\lambda\to 1-0$ and describe the effect of δ-like concentration of currents on the outer boundary of G.

First we show

Lemma 9. *For $0<\lambda<1$ sufficiently close to 1, u^λ has a unique zero (vortex) $\tilde\xi^\lambda$ and $\operatorname{dist}(\tilde\xi^\lambda,\xi^\lambda)=o(\operatorname{dist}(\xi^\lambda,\partial G))$, where ξ^λ is the unique zero of θ^λ defined through (5.21)-(5.22). Moreover, there is $\mu_0>0$ such that*

$$|u^\lambda|\ge\mu_0 \text{ in } G\setminus a_{\xi^\lambda}^{-1}(B_{1/2}(0)), \tag{7.1}$$

where a_{ξ^λ} is the conformal map given by (3.6) with $\xi=\xi^\lambda$.

Proof. Recall that $u^\lambda = e^{\tilde\varphi^\lambda/2}(\theta^\lambda + w^\lambda)$ and $\tilde\varphi^\lambda \to 0$ uniformly on $\bar G$ (cf. (5.20)). It follows from (5.30) and (i) of Lemma 3 that for $\lambda \to 1-0$ we have $|u^\lambda| \geq C(|\gamma_{\xi^\lambda}| - \varepsilon) \geq C(|a_{\xi^\lambda}| - \varepsilon) \geq C(1/2-\varepsilon)$ in $a_{\xi^\lambda}^{-1}(B_{1/2}(0))$, where C is some positive constant independent of λ. This shows (7.1).

In order to study local (in $a_{\xi^\lambda}^{-1}(B_{1/2}(0))$) behavior of u^λ we perform the rescaling $x \mapsto \zeta = a_{\xi^\lambda}(x)$. Set $U^\lambda(\zeta) = u^\lambda(a_{\xi^\lambda}^{-1}(\zeta))$, $\Theta^\lambda(\zeta) = \theta^\lambda(a_{\xi^\lambda}^{-1}(\zeta))$ and $W^\lambda(\zeta) = w^\lambda(a_{\xi^\lambda}^{-1}(\zeta))$. Note that (5.23) can be written as

$$\Delta w^\lambda = g_1^\lambda\Big(\frac{\partial\theta^\lambda}{\partial\bar z} + \frac{\partial w^\lambda}{\partial \bar z}\Big) + g_2^\lambda \mathrm{curl} A^\lambda + g_3^\lambda$$

with coefficients g_k^λ whose L^∞-norms are uniformly in λ bounded (this follows from results in Section 5, cf. (5.15), (5.20), (5.21) and the second bound in (5.8)). We will show below that the L^∞-norm of $\mathrm{curl} A^\lambda$ is also uniformly bounded. Thus we get after rescaling the above equation, for λ sufficiently close to 1,

$$|\Delta W^\lambda| \leq C_1 \mathrm{dist}(\xi^\lambda, \partial\Omega)\Big|\frac{\partial\Theta^\lambda}{\partial\bar\zeta} + \frac{\partial W^\lambda}{\partial\bar\zeta}\Big| + C_2\mathrm{dist}^2(\xi^\lambda,\partial\Omega) \text{ in } B_{3/4}(0), \quad (7.2)$$

where we have used the obvious bound $|\nabla(a_{\xi^\lambda}^{-1})| \leq C\mathrm{dist}(\xi^\lambda, \partial\Omega)$ in $B_{3/4}(0)$. The behavior of Θ^λ when $\lambda \to 1-0$ is already examined in Section 6, and we know that (up to multiplication on a constant with modulus one) $\Theta^\lambda(\zeta) \to \zeta$ in $C^k(B_{3/4}(0);\mathbb{C})$ for every $k>0$. By (5.30) we also know that $\|W^\lambda\|_{H^1(B_{3/4}(0);\mathbb{C})} \leq \|W^\lambda\|_{H^1(a_{\xi^\lambda}(G);\mathbb{C})} \to 0$ as $\lambda \to 1-0$. It follows from (7.2), by elliptic estimates, that $W^\lambda \to 0$ in $H^2_{\mathrm{loc}}(B_{3/4}(0);\mathbb{C})$. In particular, $\|W^\lambda\|_{W^{1,q}(B_{2/3}(0);\mathbb{C})} \to 0$ for every $q \geq 1$. Then (7.2) restricted to $B_{2/3}(0)$ implies that $\|W^\lambda\|_{W^{2,q}(B_{1/2}(0);\mathbb{C})} \to 0$ ($\forall q > 1$), therefore $\|W^\lambda\|_{C^1(B_{1/2}(0);\mathbb{C})} \to 0$. Thus $\Theta^\lambda + W^\lambda$ has exactly one zero in $B_{1/2}(0)$ which tends to the origin as $\lambda \to 1-0$, that is $u^\lambda = e^{\tilde\varphi^\lambda/2}(\theta^\lambda + w^\lambda)$ does have a unique zero $\tilde\xi^\lambda$ and $a_{\xi^\lambda}(\tilde\xi^\lambda) \to 0$. By an explicit computation this yields $\mathrm{dist}(\tilde\xi^\lambda, \xi^\lambda) = o(\mathrm{dist}(\xi^\lambda, \partial G))$. ☐

Lemma 10. *We have (i)* $\|h^\lambda\|_{L^\infty(G)} \leq C$, *(ii)* $\frac{\partial h^\lambda}{\partial\nu} \leq 0$ *on* $\partial\Omega$, *(iii)* h^λ *converges to zero weakly in* $H^1(G)$ *as* $\lambda \to 1-0$.

Proof. (i) We assume hereafter that the minimizer (u^λ, A^λ) is in the Coulomb gauge (2.2). Take curl in (2.5) to get the equation

$$-\Delta h = 2\frac{\partial u^\lambda}{\partial x_1} \wedge \frac{\partial u^\lambda}{\partial x_2} - \mathrm{curl}(|u^\lambda|^2 A^\lambda) \text{ in } G, \quad (7.3)$$

we also have the following boundary conditions

$$h = 0 \text{ on } \partial\Omega \text{ and } h = h^\lambda_\omega \text{ on } \partial\omega. \tag{7.4}$$

We can represent h^λ as $h^\lambda = \hat{h}^\lambda_1 + \hat{h}^\lambda_2$ with $\hat{h}^\lambda_2$ solving $\Delta\hat{h}^\lambda_2 = \mathrm{curl}(|u^\lambda|^2 A^\lambda)$ in G subject to the boundary conditions $\hat{h}^\lambda_2 = 0$ on $\partial\Omega$ and $\hat{h}^\lambda_2 = h^\lambda_\omega$ on $\partial\omega$. According to (5.3) and bound (5.8) we have $|h^\lambda_\omega| \leq C$ and $\|A^\lambda\|_{L^q(G;\mathbb{R}^2)} \leq C_q$, $\forall q \geq 1$, where C_q is independent of λ. Therefore, by elliptic estimates, the norm $\|\hat{h}^\lambda_2\|_{W^{1,q}(G)}$ is uniformly in λ bounded. This in turn implies the uniform boundedness of $\|\hat{h}^\lambda_2\|_{C(\bar{G})}$, thanks to the compactness of the embedding $W^{1,q}(G) \subset C(\bar{G})$ for $q > 2$. We next consider $\hat{h}^\lambda_1$ which satisfies $-\Delta\hat{h}^\lambda_1 = 2\frac{\partial u^\lambda}{\partial x_1} \wedge \frac{\partial u^\lambda}{\partial x_2}$ in G and zero boundary conditions on ∂G. Applying a result from [21] (see also [7]) we have $\|\hat{h}^\lambda_1\|_{H^1(G)}$, $\|\hat{h}^\lambda_1\|_{L^\infty(G)} \leq C\|u^\lambda\|^2_{H^1(G;\mathbb{C})}$, so that the required L^∞-bound follows from the fact that $\|u^\lambda\|_{H^1(G;\mathbb{C})} \leq C$ (cf. Section 5).

To demonstrate (iii) we just note that the weak convergence of h^λ follows from (5.8),(5.13) and (5.14), since we already know that $\|h^\lambda\|_{H^1(G)}$ is bounded.

To prove (ii) it suffices to show that $h^\lambda \geq 0$ in G ($h^\lambda = 0$ on $\partial\Omega$). For this purpose we derive from the pointwise equality $j^\lambda = -\nabla^\perp h$, dividing it by $|u^\lambda|$ and than taking curl, that

$$-\mathrm{div}\Big(\frac{1}{|u^\lambda|^2}\nabla h^\lambda\Big) + h^\lambda = 0 \text{ in } \{x \in G; |u^\lambda(x)| > 0\}. \tag{7.5}$$

Let $1 > \rho > 0$ be a regular value of $|u^\lambda|$ (by Sard's lemma almost all $\rho \in (0,1)$ are regular values of $|u^\lambda|$), and let x_0 be a minimum point of h^λ on the closure of $G_\rho = \{x \in G; |u^\lambda(x)| > \rho\}$. Assume by contradiction that $h^\lambda(x_0) < 0$. Then by the maximum principle applied to (7.5) the point x_0 cannot be in the interior of G_ρ. It cannot also be on $\partial\omega$, otherwise $h^\lambda(x_0) = h^\lambda_\omega < 0$ and therefore

$$\begin{aligned}\int_{\partial\omega} \frac{\partial h^\lambda}{\partial\nu}\,\mathrm{d}s = -\int_{\partial\omega} j^\lambda\cdot\tau\,\mathrm{d}s = -2\pi\mathrm{deg}(u^\lambda,\partial\omega) + \int_{\partial\omega} A^\lambda\cdot\tau\,\mathrm{d}s \\ = \int_\omega h^\lambda\,\mathrm{d}x = |\omega| h^\lambda_\omega < 0,\end{aligned}$$

thus $h^\lambda(x_0)$ is not a minimal value of h^λ in $\overline{G_\rho}$. Similar computations show that, if $|u^\lambda(x_0)| = \rho$ then

$$\frac{1}{\rho^2}\int_{|u^\lambda|=\rho} \frac{\partial h^\lambda}{\partial \nu}\,\mathrm{d}s = -2\pi + \int_{|u^\lambda|<\rho} h^\lambda \mathrm{d}x, \tag{7.6}$$

where we have used the fact that the sum of degrees of u^λ/ρ over connected components of $\partial\{x \in G; |u^\lambda(x)| < \rho\}$ is 1. Assuming $\rho \to 0$ in (7.6) we again get a contradiction, therefore $h^\lambda \geq 0$ on $\bar{G}_\rho$ when ρ is sufficiently small. Thus $h^\lambda \geq 0$ in G. □

Next we study the asymptotic behavior of currents j^λ. According to (iii) of Lemma 10, $j^\lambda \to 0$ weakly in $L^2(G;\mathbb{R}^2)$ as $\lambda \to 1-0$. One can also show that the convergence is uniform on compacts in G. Hence the currents on the boundary are of special interest to us.

Lemma 11. *Let $\xi^\lambda \to \xi^*$ ($\in \partial\Omega$, cf. Lemma 3) as $\lambda \to 1-0$, along a subsequence. Then $j^\lambda \cdot \tau \to 2\pi\delta_{\xi^*}$ in $\mathcal{D}'(\partial\Omega)$, where δ_{ξ^*} stands for the Dirac delta centered at ξ^*.*

Proof. From (ii) of Lemma 10 we know that $j^\lambda \cdot \tau \geq 0$. Hence the total variation of the measure $j^\lambda \cdot \tau\,\mathrm{d}s$ is

$$\int_{\partial\Omega} j^\lambda \cdot \tau\,\mathrm{d}s = 2\pi \mathrm{deg}(u^\lambda, \partial\Omega) - \int_\Omega h^\lambda\,\mathrm{d}x = 2\pi - \int_\Omega h^\lambda\,\mathrm{d}x,$$

and, by (iii) of Lemma 10, it tends to 2π. Therefore it suffices to show that

$$\int_{\partial\Omega} \Phi j^\lambda \cdot \tau\,\mathrm{d}s \to 0 \quad \forall \Phi \in C^1(\partial\Omega) \text{ such that } \Phi = 0 \text{ in a neighborhood of } \xi^*.$$

Let Φ be extended into G to have $\Phi \in C^1(\bar{G})$, $\Phi = 0$ on $\partial\omega$ and in $G \cap B_\rho(\xi^*)$ for some $\rho > 0$. Assume that λ is so close to 1 that $a^{-1}_{\xi^\lambda}(B_{1/2}(0)) \subset B_\rho(\xi^*)$, then, by Lemma 9, $|u^\lambda| \geq \mu_0 > 0$ in $G \setminus B_\rho(\xi^*)$. Multiply (7.5) by Φ to get after integrating over $G \setminus B_\rho(\xi^*)$,

$$-\int_{\partial\Omega} \Phi j^\lambda \cdot \tau\,\mathrm{d}s = \int_{\partial\Omega} \Phi \frac{\partial h^\lambda}{\partial \nu}\,\mathrm{d}s = \int_G \Big(\frac{1}{|u^\lambda|^2}\nabla\Phi \cdot \nabla h^\lambda + h^\lambda \Phi\Big)\,\mathrm{d}x.$$

The right hand side of this equality tends to zero as $\lambda \to 1-0$, since $\frac{1}{|u^\lambda|^2}\nabla\Phi \to \nabla\Phi$ strongly in $L^2(G;\mathbb{R}^2)$, while $h^\lambda \to 0$ weakly in $H^1(G)$. □

Remark 3. *A reasoning similar to the proof of Lemma 11 shows that $j^\lambda \to 0$ in $D'(\partial\omega)$ as $\lambda \to 1-0$. (This is due to the fact that u^λ has a unique vortex approaching $\partial\Omega$ in the limit.)*

8. Explicit formula for energy bounds

The right hand side $I(\xi,\lambda)$ in the upper bound (3.14) can be equivalently rewritten as

$$I(\xi,\lambda)=\pi+\frac{1}{8K_G}\Big(\int_{\partial\omega}\frac{\partial\phi_\xi}{\partial\nu}\,\mathrm{d}s\Big)^2-\frac{1-\lambda}{8}\int_G(|a_\xi|^2e^{\phi_\xi}-1)^2\mathrm{d}x, \tag{8.1}$$

where ϕ is the unique solution of

$$\begin{cases} -\Delta\phi_\xi=1-|a_\xi(z)|^2e^{\phi_\xi}\text{ in }G\\ \phi_\xi=0\text{ on }\partial\Omega\\ \phi_\xi=-\log|a_\xi(z)|^2\text{ on }\partial\omega,\end{cases} \tag{8.2}$$

$a_\xi(z)$ is given by (3.6) with $\mathcal{F}$ being a fixed conformal map from Ω onto the unit disk. Indeed, the solution φ_ξ of (3.3)-(3.4) and ϕ_ξ are related by $\varphi_\xi=\phi_\xi-\sigma_\xi$, where $\sigma_\xi=\log|\gamma_\xi/a_\xi|$ ($\Delta\sigma_\xi=0$ in G). Note also that

$$\int_{\partial\omega}\frac{\partial\varphi_\xi}{\partial\nu}\,\mathrm{d}s=\int_{\partial\omega}\frac{\partial\phi_\xi}{\partial\nu}\,\mathrm{d}s,$$

since σ_ξ satisfies (3.7).

The following lemma proves the explicit asymptotic formula (3.15).

Lemma 12. *Let $\xi\in G$ and let $\xi^*=\xi^*(\xi)\in\partial\Omega$ to be the nearest point projection of ξ on $\partial\Omega$. Then for sufficiently small $\delta=dist(\xi,\partial\Omega)$*

(i) $\displaystyle\int_{\partial w}\frac{\partial\phi_\xi}{\partial\nu}\,\mathrm{d}s=4\pi\delta\frac{\partial V}{\partial\nu}(\xi^*)+o(\delta)$, *where V is the solution of (1.4),*

(ii) $\displaystyle\int_G(|a_\xi|^2e^{\phi_\xi}-1)^2\,\mathrm{d}x=8\pi\delta^2|\log\delta|+O(\delta^2)$.

In the proof of both statements of Lemma 12 we will make use of the following formulas, as $\xi\to\partial\Omega$

$$\int_G\big(1-|a_\xi|^2\big)^2\mathrm{d}x=8\pi\delta^2(|\log\delta|+O(1)), \tag{8.3}$$

$$\int_G\big(1-|a_\xi|^2\big)\mathrm{d}x=O(\delta), \tag{8.4}$$

where δ is the distance from ξ to $\partial\Omega$. We postpone the proof of these formulas and proceed to the

Proof of (i) of Lemma 12. We first show that

$$\int_{\partial\omega} \frac{\partial \phi_\xi}{\partial \nu} \mathrm{d}s = \int_{\partial\omega} \frac{\partial \phi^*_\xi}{\partial \nu} \mathrm{d}s + o(\delta) \text{ as } \delta \to 0, \tag{8.5}$$

where ϕ^*_ξ is the unique solution of the auxiliary linear problem

$$\begin{cases} -\Delta\phi^*_\xi + \phi^*_\xi = 1 - |a_\xi(z)|^2 \text{ in } G \\ \phi^*_\xi = 0 \text{ on } \partial\Omega \\ \phi^*_\xi = -\log|a_\xi(z)|^2 \text{ on } \partial\omega. \end{cases} \tag{8.6}$$

We claim that

$$\|\phi_\xi\|_{C(\bar{G})} < C\delta \text{ as } \delta \to 0. \tag{8.7}$$

This implies the bound

$$\begin{aligned} \|-\Delta(\phi_\xi - \phi^*_\xi) + (\phi_\xi - \phi^*_\xi)\|_{L^2(G)} &= \|\phi_\xi(1 - |a_\xi|^2) + |a_\xi|^2(1 - e^{\phi_\xi} + \phi_\xi)\|_{L^2(G)} \\ &\leq \delta\|1 - |a_\xi|^2\|_{L^2(G)} + C\delta^2 \leq C_1\delta^2(|\log\delta| + 1), \end{aligned}$$

where we have used (8.3). Since $\phi_\xi = \phi^*_\xi$ on ∂G, we can easily derive (8.5) by standard elliptic estimates.

Proof of Claim (8.7). Due to (8.2) we have $-\Delta(\phi_\xi + \log|a_\xi(z)|^2) = 1 - |a_\xi(z)|^2 e^{\phi_\xi}$ in $G \setminus B_r(\xi)$ for every $r > 0$. By applying the maximum principle to this equation, we conclude that $\phi_\xi \leq -\log|a_\xi(z)|^2$ in G. The latter inequality implies that $1 - |a_\xi(z)|^2 e^{\phi_\xi} \geq 0$. Hence, we can apply the maximum principle once more to (8.2) to conclude that $\phi_\xi \geq 0$ in G. Thus, $0 \leq 1 - |a_\xi(z)|^2 e^{\phi_\xi} \leq 1 - |a_\xi(z)|^2$. On the other hand, a direct computation shows that

$$1 - |a_\xi(z)|^2 = (|\mathcal{F}(\xi)|^2 - 1)\frac{|\mathcal{F}(z)|^2 - 1}{|\mathcal{F}(z)\overline{\mathcal{F}(\xi)} - 1|^2} \leq C\frac{\delta}{|z - \xi^*|} \text{ as } \delta \to 0. \tag{8.8}$$

This allows us to estimate L^p-norm of the right hand side of (8.2) for every $1 < p < 2$ and next obtain, by standard elliptic estimates, that

$$\|\phi_\xi\|_{W^{2,p}(G)} \leq C(p)\delta \text{ as } \delta \to 0.$$

By using the Sobolev embedding the required result (8.7) follows. □

Remark 4. *In the proof of Claim (8.7) we showed that $|a_\xi|e^{\phi_\xi} \leq 1$ in G, therefore $|\gamma_\xi|e^{\varphi_\xi} \leq 1$ in G (since $\varphi_\xi = \phi_\xi - \log|\gamma_\xi/a_\xi|$).*

Proof of (i) of Lemma 12 completed. Now multiply the equation in (8.6) by V (the unique solution of problem (1.4)) and integrate by parts to get

$$\int_{\partial\omega} \frac{\partial \phi^*_\xi}{\partial\nu} \mathrm{d}s = \int_G (1 - |a_\xi|^2)V - \int_{\partial\omega} \log|a_\xi|^2 \frac{\partial V}{\partial \nu} \mathrm{d}s \tag{8.9}$$

On the other hand, since $\Delta \log|a_\xi|^2 = 4\pi\delta_\xi(x)$ in Ω and $\log|a_\xi|^2 = 0$ on $\partial\Omega$, we have

$$4\pi V(\xi) = \int_G \log|a_\xi|^2 V \mathrm{d}x + \int_{\partial\omega} \log|a_\xi|^2 \frac{\partial V}{\partial\nu}\mathrm{d}s \tag{8.10}$$

By adding (8.9) to (8.10), we obtain

$$\int_{\partial\omega} \frac{\partial\phi^*_\xi}{\partial\nu}\mathrm{d}s = 4\pi(V(\xi^*) - V(\xi)) + \int_G (\log|a_\xi|^2 + 1 - |a_\xi|^2)V\mathrm{d}x.$$

Thus, in view of (8.5) it remains only to show that the last term in the above equality is of order $o(\delta)$. To this end we split it as

$$\int_G (\log|a_\xi|^2 + 1 - |a_\xi|^2)V\mathrm{d}x = \int_{G\setminus B_{\sqrt{\delta}}(\xi^*)} \cdots + \int_{G\cap B_{\sqrt{\delta}}(\xi^*)} \cdots =: I_1 + I_2.$$

According to (8.8) we have $|\log(1 - 1 + |a_\xi|^2) + 1 - |a_\xi|^2| \leq C\delta^2/|x - \xi^*|^2$ in $G \setminus B_{\sqrt{\delta}}(\xi^*)$, therefore $I_1 = O(\delta^2)$ (note that $|V(x)| \leq C|x - \xi^*|$); while $I_2 = O(\delta^{3/2}|\log\delta|)$, that can be verified by using the obvious bound $|\log|a_\xi|^2 + 1 - |a_\xi|^2| \leq C(|\log|x - \xi|| + 1)$. Statement (i) is proved. □

Proof of (ii) of Lemma 12. Integrating the identity

$$(|a_\xi|^2 e^{\phi_\xi} - 1)^2 = (|a_\xi|^2 - 1)^2 + 2|a_\xi|^2(e^{\phi_\xi} - 1)(|a_\xi|^2 - 1) + |a_\xi|^4(e^{\phi_\xi} - 1)^2$$

over G we use estimates (8.3), (8.4), (8.7) and the Cauchy-Schwarz inequality to establish the following

$$\int_G (|a_\xi(z)|^2 e^\varphi - 1)^2 dx = \int_G (|a(z,\xi)|^2 - 1)^2 dx + O(\delta^2) = 8\pi\delta^2|\log\delta| + O(\delta^2).$$

Thus Lemma 12 is completely proved. □

Calculation of (8.3) and (8.4). For brevity we show only (8.3) (the demonstration of (8.4) follows the same lines). Perform the change of variables $\zeta = \mathcal{F}(x)$ to get, after simple computations,

$$\int_G \big(1 - |a_\xi|^2\big)^2 \mathrm{d}x = \int_{B_1(0)\setminus\mathcal{F}(\omega)} \big(1 - |m_{\mathcal{F}(\xi)}|^2\big)^2 \frac{d\zeta}{\mathrm{Jac}\mathcal{F}(\zeta)}$$
$$= \frac{1}{\mathrm{Jac}\mathcal{F}(\xi)} \int_{B_1(0)} \Big(1 - \big|m_{\mathcal{F}(\xi)}(\zeta)\big|^2\Big)^2 d\zeta + O(\delta^2) \quad (8.11)$$

where $m_\mu(z) = (z-\mu)/(\bar{\mu}z - 1)$ is the classical Möbius conformal map from the unit disk onto itself. The integral in the right hand side of (8.11) can be calculated explicitly. We obtain, by using the coarea formula twice,

$$\int_{B_1(0)} (|m_\mu(x)|^2 - 1)^2 \mathrm{d}x = \int_0^1 \mathrm{d}t \int_{|m_\mu(x)|=t} (t^2-1)^2 \frac{\mathrm{d}\mathcal{H}^1}{|\nabla |m_\mu(x)||}$$
$$= \int_0^1 (t^2-1)^2 \mathrm{d} \int_0^t \mathrm{d}\tau \int_{|m_\mu(x)|=\tau} \frac{\mathrm{d}\mathcal{H}^1}{|\nabla |m_\mu(x)||}$$
$$= \int_0^1 (t^2-1)^2 \mathrm{d}\big(\mathrm{area}(m_\mu^{-1}(B_t(0)))\big).$$

Note that the inverse conformal map $m_\mu^{-1}(z)$ coincides with $m_\mu(z)$. Moreover, the inverse image $m_\xi^{-1}(B_t(0))$ of the disk $B_t(0)$ is the disk $B_r(y)$ with the radius $r = t(1-|\mu|^2)/(1-t^2|\mu|^2)$ and the center at $y = \mu(1-t^2)/(1-t^2|\mu|^2)$. Therefore we get, after integrating by parts,

$$\int_{B_1(0)} (|m_\mu(x)|^2 - 1)^2 \mathrm{d}x = 2(1-|\mu|^2)^2 \pi \int_0^1 \frac{(1-t^2)t^2 \mathrm{d}t^2}{(1-t^2|\mu|^2)^2},$$

and elementary calculations lead to the following asymptotic formula, as $|\mu| \to 1 - 0$

$$\int_{B_1(0)} \big(1 - |m_\mu(\zeta)|^2\big)^2 \mathrm{d}\zeta = 2\pi(1-|\mu|^2)^2(|\log(1-|\mu|^2)| + O(1)). \quad (8.12)$$

Finally, by the conformality of $\mathcal{F}$ we have

$$(1 - |\mathcal{F}(\xi)|^2)^2 = 4\delta^2 \mathrm{Jac}\mathcal{F}(\xi) + O(\delta^3). \quad (8.13)$$

Thus (8.11), (8.12) and (8.13) yield (8.3). □

Lemma 12 allows us to rewrite the lower bound (5.1) of Lemma 2 in the form

$$m(\lambda) = F_\lambda[u^\lambda, A^\lambda] \geq \pi + \frac{2\pi^2}{K_G}\delta^2\Big|\frac{\partial V}{\partial \nu}(\hat{\xi}^\lambda)\Big|^2 - \pi\delta^2(1-\lambda)|\log\delta|(1+o(1)) + o(\delta^2), \tag{8.14}$$

as $\lambda \to 1-0$, where $\hat{\xi}^\lambda$ is the nearest point projection on $\partial\Omega$ of the point ξ^λ and $\delta = \text{dist}(\xi^\lambda, \partial\Omega)$ ($\delta = \delta(\lambda) \to 0$). Recall that the point $\xi^\lambda \in G$ was defined in Section 5 as the unique zero of the auxiliary map θ^λ constructed by means of u^λ and A^λ. By Lemma 9 we can redefine ξ^λ as the unique zero (vortex) of u^λ so that (8.14) remains valid. On the other hand, by (3.14), (8.1) and Lemma 12,

$$m(\lambda) \leq I(\xi, \lambda) = \pi + \frac{2\pi^2}{K_G}\tilde{\delta}^2\Big|\frac{\partial V}{\partial \nu}(\hat{\xi})\Big|^2 - \pi(1-\lambda)(\tilde{\delta}^2|\log\tilde{\delta}| + O(\tilde{\delta}^2)) + o(\tilde{\delta}^2), \tag{8.15}$$

where $\hat{\xi}$ is an arbitrary point on $\partial\Omega$ and $\tilde{\delta} > 0$ is a small parameter ($\hat{\xi}$ is the nearest point projection on $\partial\Omega$ of $\xi \in G$ and $\tilde{\delta} = \text{dist}(\xi, \partial\Omega)$). It follows from (8.14) and (8.15) that, as $\lambda \to 1-0$

(a) $\delta = \exp\big(\frac{-2\pi}{(1-\lambda)K_G}|\frac{\partial V}{\partial \nu}(\hat{\xi}^\lambda)|^2(1+o(1))\big)$;

(b) $|\frac{\partial V}{\partial \nu}(\hat{\xi}^\lambda)|^2 \to M_G$, where $M_G = \min\{|\frac{\partial V}{\partial \nu}(\hat{\xi})|^2; \hat{\xi} \in \partial\Omega\} (> 0)$.

Thus we have, in particular, that the unique zero (vortex) of u^λ converges (up to extracting a subsequence) to a point minimizing $|\frac{\partial V}{\partial \nu}|^2$ on $\partial\Omega$. This completes the proof of Theorem 1.

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