Necessary conditions for existence of local Ginzburg-Landau minimizers with prescribed degrees on the boundary

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Abstract

We study the minimization problem for simplified Ginzburg-Landau functional in doubly connected domain. This minimization problem is a subject to "semi-stiff" boundary conditions: $|u| = 1$ and prescribed degrees p and q on the outer and inner boundaries respectively. Following the work of L. Berlyand and V.Rybalko [7], we additionally prescribe the degree in the bulk (approximate bulk degree) to be d. The work [7] established the *sufficient* conditions on the existence of Ginzburg-Landau minimizers, given in terms of p, q and d. The present work complements the result of [7] by providing the *necessary* conditions for the existence of nontrivial (nonconstant) minimizers.

Contents

1 Introduction and main results

Consider the minimization problem for the simplified Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_G \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2 \right\}, \tag{1}$$

in a smooth doubly connected domain $G = \Omega \setminus \omega \subset \mathbb{C}$. Note that any critical point of (1) satisfies the Ginzburg-Landau equation in G:

$$-\Delta u = \frac{1}{\varepsilon^2}u(1-|u|^2) \tag{2}$$

The solutions of (2) with isolated zeros (vortices) are of special importance since they model the observable physical states during phase transitions in superconductors.

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Minimization problems for Ginzburg-Landau type functionals have been extensively studied by a variety of authors. The pioneering work on modeling Ginzburg-Landau vortices is the work of Bethuel, Brezis and Hélein [8]. In this work the authors suggested to consider a simplified Ginzburg-Landau model (1), in which the physical source of vortices, the external magnetic field, is modeled via a Dirichlet boundary condition with a positive degree on the boundary. The analysis of full Ginzburg-Landau functional, with induced and applied magnetic fields, was later performed by Sandier and Serfaty in [14].

This work addresses the issue of the existence of local minimizers of (1) in the class of functions with "semi-stiff" boundary conditions

$$\mathcal{J} := \{u \in H^1(G, \mathbb{C}) \,|\, |u| = 1 \text{ a.e. on } \partial G\}. \tag{3}$$

The global minimization problem (1)-(3) gives only trivial constant minimizers, e.g. $u \equiv 1$. In order to find the critical points of (1) with vortices, it is natural to impose the degrees (winding numbers) q and p on $\partial\Omega$ and $\partial\omega$ correspondingly. This leads us to the minimization problem for (1) in the class

$$\mathcal{J}_{pq} := \{u \in \mathcal{J}, \deg(u, \partial\omega) = p, \deg(u, \partial\Omega) = q\} \tag{4}$$

Here, the topological degree is given by

$$\deg(u, \gamma) := \frac{1}{2\pi} \int_\gamma u \times \frac{\partial u}{\partial \tau} \mathrm{d}\sigma \tag{5}$$

where γ is a smooth closed curve, $u \in C^1(\gamma, \mathbb{S}^1)$ and $a \times b := \frac{i}{2}(a\bar{b} - b\bar{a})$ for all $a, b \in \mathbb{C}$. It is worth noting that the degree is a well-defined integer for $u \in H^{1/2}(\gamma, \mathbb{S}^1)$ [?].

The minimization problem for (1) in (4) has been studied in [4], [10], [5], [6] [3], [7], [9] and others. The works [13], [12] and [1] studied the minimization problem for full Ginzburg-Landau functional (with induced magnetic field) in class (4). As observed in [4] and [5], the class $\mathcal{J}_{pq}$ is not weakly H^1 closed since the degree at the boundary may change in the weak H^1 limit. Therefore, it is not difficult to see ([5] and [7]) that if $p \neq q$, one can construct an explicit minimizing sequence for (1) in (4), whose weak H^1 limit does not belong to $\mathcal{J}_{pq}$, which implies that there are no global minimizers of (1) in (4) for $p \neq q$.

The case $p = q = 1$ was studied in [10], [5] and [3]. The existence of vortexless minimizers was established in sufficiently thin domains. It was also shown in [3] that for sufficiently small ε there are no global minimizers of (1) in $\mathcal{J}_{11}$ in thick domains.

Due to the lack of weak H^1 compactness of $\mathcal{J}_{pq}$, the question of existence of solutions of (2) in $\mathcal{J}_{pq}$ is highly nontrivial. The work [7] provided the answer to this question and established the existence of such solutions for small ε. Moreover, the solutions in [7] are stable in the sense that they are *local* minimizers of (1) in $\mathcal{J}$. The work [9] generalized the results in [7] for the case of a multiply connected domain. The main tool in establishing the stable solutions was the *approximate bulk degree.*

Definition 1. *[7] Let $u \in H^1(G, \mathbb{C})$ and $V \in C^\infty(G, \mathbb{R})$ be a solution to the scalar boundary value problem*

$$\begin{cases} \Delta V = 0 \text{ in } G; \\ V = 1 \text{ on } \partial\Omega; \\ V = 0 \text{ on } \partial\omega. \end{cases} \tag{6}$$

Then the approximate bulk degree of u in G is the following scalar quantity:

$$\mathrm{abdeg}(u) = \frac{1}{2\pi} \int_G u \times (\partial_{x_1} V \, \partial_{x_2} u - \partial_{x_2} V \, \partial_{x_1} u) \, \mathrm{d}x. \tag{7}$$

The key property of abdeg(u) is that, unlike the degree, it is preserved in weak- H^1 limits, i.e. abdeg$(u_n) \to$ abdeg(u) if $u_n \rightharpoonup u$ in $H^1(G)$, $n \to \infty$.

The construction of the solutions of (1) in $\mathcal{J}$ was based on the study of the following minimization problem

$$m_\varepsilon(p, q, d) := \inf\{E_\varepsilon(u);\ u \in \mathcal{J}_{pq}^{(d)}\}, \tag{8}$$

where

$$\mathcal{J}_{pq}^{(d)} = \{u \in \mathcal{J}_{pq}; d - 1/2 \leq \text{abdeg}(u) \leq d + 1/2\}. \tag{9}$$

The following existence result holds ([7]):

Theorem 1. *For any integers p, q and $d > 0$ $(d < 0)$ with $d \geq \max\{p, q\}$ $(d \leq \min\{p, q\})$ there exists $\varepsilon_1 = \varepsilon_1(p, q, d) > 0$ such that the infimum in (8) is always attained, when $\varepsilon < \varepsilon_1$. Additionally, each minimizer lies in $\mathcal{J}_{pq}^{(d)}$ with its open neighborhood, therefore, the minimizers in $\mathcal{J}_{pq}^{(d)}$ are distinct local minimizers in $\mathcal{J}$.*

It was conjectured in [7] that the condition $d \geq \max\{p, q\}$ $(d \leq \min\{p, q\})$ is essential and in the contrary case there is a threshold value ε_0 s.t. for $\varepsilon < \varepsilon_0$ the minimizer in the minimization problem (8) is not attained. The proof of this conjecture is the main objective of this paper.

Our main results are:

Theorem 2. *Let $d > 0$, $d \leq \min\{p, q\}$ (or $d < 0$, $d \geq \max\{p, q\}$) and either $p \neq d$ or $q \neq d$. Let u be a weak limit of a minimizing sequence for problem (8) (such a minimizing sequence always exists). Then $u \notin \mathcal{J}_{pq}^{(d)}$ when ε is sufficiently small.*

Theorem 3. *Let u be a weak limit of a minimizing sequence for problem (8) with $d = 0$ and $p, q \in \mathbb{Z}$, $p \neq 0$ or $q \neq 0$. Then $u \notin \mathcal{J}_{pq}^{(d)}$ when ε is sufficiently small.*

Theorems 2 and 3 provide the necessary conditions for the existence of GL minimizers by imposing natural restrictions on the bulk degrees of stable GL solutions. These results also complement the Theorem 1 in the sense that they show that the conditions $d > 0$ $(d < 0)$ with $d \geq \max\{p, q\}$ $(d \leq \min\{p, q\})$ are essential rather than technical.

Theorem 3 is a generalization of the main result of [3], which was proved for $p = q = 1$. The main idea of Theorem 3 may be summarized as follows. We assume that the minimizer $u_\varepsilon \in \mathcal{J}_{pq}^{(0)}$ exists. Then, by a clever choice of test functions we may conclude that $E_\varepsilon[u_\varepsilon] \leq \pi(|p| + |q|)$. Furthermore, certain properties of GL minimizers with zero bulk degree, such as Proposition 4, enable us to construct an auxiliary quadratic functional S_ε s.t.

$$E_\varepsilon[u_\varepsilon] \geq S_\varepsilon[u_\varepsilon] := \frac{1}{2}\int_G |\nabla u_\varepsilon|^2 + \frac{1}{2}\int_{G_\delta}\left[\frac{1}{\varepsilon^2}(Re(u_\varepsilon) - 1)^2 - \varepsilon^2(Im(u_\varepsilon))^2\right]$$

for some $G_\delta \subset\subset G$. Minimization of the quadratic functional S_ε in $\mathcal{J}_{pq}^{(0)}$ leads to linear Euler-Lagrange equations, which may be solved explicitly using the separation of variables and Fourier series. This enables us to establish that $S_\varepsilon[u_\varepsilon] > \pi(|p| + |q|)$ and we arrive at contradiction with the existing upper bound.

In order to prove Theorem 2, we again assume that the minimizer $u_\varepsilon \in \mathcal{J}_{pq}^{(d)}$ exists. The above approach, however, cannot be extended to the proof of Theorem 2, because if $d \neq 0$ the estimates of Proposition 4 are no longer valid. Therefore, we use a variant of Substitution Lemma due to Lassoued and Mironescu [11]. For $u \in \mathcal{J}_{pq}^{(d)}$ we write $u = u^{(d)}v$, where $u^{(d)}$ is a minimizer of E_ε in $\mathcal{J}_{d,d}^{(d)}$ (its existence was established in [7]) and $v_\varepsilon \in \mathcal{J}_{p-d,q-d}^{(0)}$. Then

$$E_\varepsilon[u_\varepsilon] = E_\varepsilon[u^{(d)}] + M_\varepsilon[v_\varepsilon] + \frac{1}{\varepsilon^2}\int_G |u^{(d)}|^4(1 - |v_\varepsilon|^2)^2$$

where

$$M_\varepsilon[v] \approx \frac{1}{2}\int_G |\nabla v|^2 + \pi abdeg(v) \tag{10}$$

The minimization problem for $M_\varepsilon[v]$ in $\mathcal{J}^{(0)}_{p-d,q-d}$ requires subtle techniques of estimating Fourier coefficients, which are essentially different from the ones in [3]. The derivation of the sharp lower bounds for the problems of type (10), based on the estimates of Fourier coefficients, is the main technical novelty of the present work. We establish that, for sufficiently small ε, $M_\varepsilon[v] > \pi(|p-d|+|q-d|)$ for all $v \in \mathcal{J}^{(0)}_{p-d,q-d}$. This contradicts the existing upper bound $\inf_{v\in\mathcal{J}^{(0)}_{p-d,q-d}} M_\varepsilon[v] \leq \pi(|p-d|+|q-d|)$, which may be obtained by a thorough choice of a test function, and thus completes the proof of Theorem 2.

2 Proof of Theorem 2: Energy decomposition

Without loss of generality, let $d > 0$, and the integers p and q are such that $p > d$ and $q \geq d$. Consider the auxiliary minimization problem

$$I_0(d,G) := \inf\{\int_G |\nabla u|^2 \mathrm{d}x;\ u \in H^1(G;\mathbb{S}^1),\ \deg(u,\partial\omega) = \deg(u,\partial\Omega) = d\} \tag{11}$$

Proposition 1. *[8] There exists a unique (up to multiplication by constants with unit modulus) solution u_∞ of the minimization problem (11), and u_∞ is a regular harmonic map in G (i.e. $-\Delta u_\infty = u_\infty|\nabla u_\infty|^2$ in G, $u_\infty \in H^1(G)$) satisfying $u_\infty \times \frac{\partial u_\infty}{\partial\nu} = 0$ on ∂G.*

Let $u^{(d)} = u^{(d)}_\varepsilon$ be a minimizer of 1 in $\mathcal{J}^{(d)}_{d,d}$ (its existence is established in [7], Lemma 19).

Lemma 1. *There exists $\alpha_\varepsilon \in \mathbb{S}^1$ s.t.*

$$\alpha_\varepsilon u^{(d)} \to u_\infty, \varepsilon \to 0 \tag{12}$$

in $C^{1,\beta}(\bar{G})$, $\beta < 1$, where u_∞ is a minimizer of (11).

Lemma 1 was proved in [5], Corollary 8.2 for $d = 1$, its proof may be easily adapted for an arbitrary d. This Lemma implies that the minimizer $u^{(d)}$ is vortexless for sufficiently small ε. Thus we can write $u^{(d)} = \rho e^{id\theta}$, where ρ, $e^{i\theta}$ and $\nabla\theta$ are smooth maps defined globally on G. Following [7], we introduce the change of coordinates

$$\Psi : (x,y) \to (h,\theta) \text{ in } \bar{G}, \tag{13}$$

where θ solves

$$\begin{cases} \operatorname{div}(\rho^2\nabla\theta) = 0 \text{ in } G; \\ \frac{\partial\theta}{\partial\nu} = 0 \text{ on } \partial G, \end{cases} \tag{14}$$

and h solves

$$\begin{cases} \nabla^\perp h = (u^{(d)} \times \partial_{x_1} u^{(d)}, u^{(d)} \times \partial_{x_2} u^{(d)}) \text{ in } G; \\ h = 0 \text{ on } \partial\Omega. \end{cases} \tag{15}$$

Note that it follows from (15) that

$$\nabla h = -d\rho^2\nabla^\perp\theta. \tag{16}$$

The problem (15) has a unique solution h (see [7], Lemma 7). Moreover, h is constant on $\partial\omega$ and

$$h_\varepsilon := h(\partial\omega) = -2\pi\frac{\mathrm{abdeg}(u)}{\mathrm{cap}(G)} \tag{17}$$

where $\text{abdeg}(u)$ is given by (7) and

$$\text{cap}(G) := \int_G |\nabla V|^2 dx, \quad V \text{ solves (6)}$$

is H^1 - *capacity* ("measure of thickness") of G. For example, if $G = B_R(0) \setminus B_r(0)$, $\text{cap}(G) = \frac{2\pi}{\ln(R/r)}$.

Lemma 2. *The map Ψ, given by (13), is a C^1 diffeomorphism on $\bar{G}$.*

Proof. It follows from [5], Lemma 4.4, that $u^{(d)} \in C^\infty(\bar{G})$. Therefore ρ and $\nabla\theta \in C^\infty(\bar{G})$, and due to (16), $\nabla h \in C^\infty(\bar{G})$. It remains to show that $Jac(\Psi) \neq 0$ in $\bar{G}$. Using (16),

$$Jac(\Psi) = d\rho^2 |\nabla\theta|^2 \tag{18}$$

On the other hand, by Lemma 1, we have $d^2\rho^2|\nabla\theta|^2 \to |\nabla\theta_\infty|^2$ in $C(\bar{G})$ as $\varepsilon \to 0$, where $u_\infty = e^{i\theta_\infty}$ minimizes (11). Note that θ_∞ solves

$$\begin{cases} \Delta\theta_\infty = 0 \text{ in } G; \\ \frac{\partial\theta_\infty}{\partial\nu} = 0 \text{ on } \partial G; \\ \int_{\partial\Omega} \frac{\partial\theta_\infty}{\partial\tau} ds = \int_{\partial\omega} \frac{\partial\theta_\infty}{\partial\tau} ds = d. \end{cases} \tag{19}$$

Let $\mathcal{F} : G \to B$ be a conformal map between G and an annulus $B := B_1(0) \setminus B_r(0)$, and $\tilde{u} = e^{i\tilde{\theta}_\infty}$ be the minimizer of (11) in B. The equation (19) in B admits an explicit multivalued solution $\tilde{\theta}_\infty = \arg\left(\frac{z^d}{|z|^d}\right)$. The conformal invariance of the Dirichlet integral implies the following relation

$$|\nabla\tilde{\theta}_\infty(z)|^2 = |\nabla\theta_\infty(\mathcal{F}^{-1}(z))|^2 |\nabla\mathcal{F}^{-1}(z)|^2, \ z \in B,$$

and thus $|\nabla\theta_\infty(\mathcal{F}^{-1}(z))|^2 \neq 0, \ z \in B$. It follows from (18) that $Jac(\Psi) \neq 0$ in $\bar{G}$, which yields the desired result. □

We now proceed with an energy decomposition for E_ε. For any $\tilde{w} \in \mathcal{J}^{(d)}_{p,q}$, write $\tilde{w} = \rho w$, $w \in \mathcal{J}^{(d)}_{p,q}$. By Lemma 21 [7], we have

$$E_\varepsilon[\tilde{w}] = E_\varepsilon[u^{(d)}] + H_\varepsilon[w], \tag{20}$$

where

$$H_\varepsilon[w] := \frac{1}{2}\int_G \rho^2|\nabla w|^2 \mathrm{d}x - \frac{d^2}{2}\int_G |\nabla\theta|^2\rho^2|w|^2 \mathrm{d}x + \frac{1}{4\varepsilon^2}\int_G \rho^4(|w|^2-1)^2 \mathrm{d}x. \tag{21}$$

We proceed further by factoring $w \in \mathcal{J}^{(d)}_{p,q}$ as $w = e^{id\theta}v$, $v = v_1 + iv_2 \in \mathcal{J}^{(0)}_{p-d,q-d}$ (such representation is clearly valid for any $w \in \mathcal{J}^{(d)}_{p,q}$).

$$H_\varepsilon[w] \equiv F_\varepsilon[v] := \frac{1}{2}\int_G \rho^2|\nabla v|^2 + d\int_G \rho^2(v_1\nabla v_2\cdot\nabla\theta - v_2\nabla v_1\cdot\nabla\theta) + \frac{1}{4\varepsilon^2}\int_G \rho^4(1-|v|^2)^2 \tag{22}$$

The following proposition is crucial in establishing Theorem 2.

Proposition 2. *Fix $\varepsilon > 0$ and assume $F_\varepsilon[v] > \pi((p-d)+(q-d))$ for any $v \in \mathcal{J}^{(0)}_{p-d,q-d}$. Then*

$$\inf_{v \in \mathcal{J}^{(0)}_{p-d,q-d}} F_\varepsilon[v] \tag{23}$$

is not attained.

Proof. To simplify the presentation, assume $p > q$ and $q = d$ (the general case is treated analogously). By, contradiction, assume $v^* \in \mathcal{J}^{(0)}_{p-d,0}$ is a minimizer of (23). Then

$$F_\varepsilon[v^*] = \pi(p-d) + \delta \tag{24}$$

for some $\delta > 0$. On the other hand, we are going to explicitly construct a test function $\eta \in \mathcal{J}^{(0)}_{p-d,0}$ s.t.

$$F_\varepsilon[\eta] \leq \pi(p-d) + \frac{\delta}{2} \tag{25}$$

which would contradict the minimality of v^*. Let $\mathcal{F} : G \to B_1(0) \setminus B_r(0)$ be a fixed conformal map as in Lemma 2. For $\xi \in G$ define

$$a(z,\xi) := \left(\frac{\mathcal{F}(\xi) - \mathcal{F}(z)}{1 - \overline{\mathcal{F}(\xi)}\mathcal{F}(z)} \right)^{p-d}$$

Note that

$$a(z,\xi) \to^w 1 \text{ in } H^1(G) \text{ as } \xi \to \partial\Omega \text{ and } a(z,\xi) \to 1 \text{ in } C^\infty(K) \text{ for any compact } K \subset (G \cup \partial\omega). \tag{26}$$

However, $a(z,\xi) \notin \mathcal{J}^{(0)}_{p-d,0}$ because $|a(z,\xi)| \neq 1$ for $z \in \partial\omega$. We first modify a in order to have constant modulus on $\partial\omega$ (see [12]). Consider

$$\gamma(z,\xi) := a(z,\xi) e^{f_\xi(z) + i g_\xi(z)}$$

where the real-valued function f_ξ satisfies

$$\begin{cases} \Delta f = 0 \text{ in } G; \\ f = 0 \text{ on } \partial\Omega; \\ f = c_\xi - \ln|a(z,\xi)| \text{ on } \partial\omega, \ c_\xi - \text{ constant}; \\ \int_{\partial\omega} \frac{\partial f}{\partial\nu} ds = 0 \end{cases} \tag{27}$$

and g_ξ is a single-valued harmonic conjugate of f_ξ (which exists due to the last condition in (27)). The resulting function $\gamma(z,\xi)$ is analytic in G and satisfies $|\gamma| = 1$ on $\partial\Omega$ and $|\gamma| = e^{c_\xi}$ on $\partial\omega$. We may now define

$$\eta(z,\xi) := \gamma(z,\xi) b(z,\xi) \in \mathcal{J}^{(0)}_{p-d,0} \tag{28}$$

where $b(z,\xi) = (e^{-c_\xi} - 1)(1 - V(z)) + 1$ and V solves (6). It follows from (27) that $c_\xi \to 0$ as $\xi \to \partial\Omega$, thus $b(z,\xi) \to 1$ in $C^\infty(\bar{G})$, $\xi \to \partial\Omega$. Thus, η satisfies (26) as well.

In order to proceed, rewrite F using the pointwise identities

$$\frac{1}{2}|\nabla v|^2 \equiv \text{Jac}v + 2\left|\frac{dv}{d\bar{z}}\right|^2 = \text{div}(v \times v_{x_2}, v_{x_1} \times v) + 2\left|\frac{dv}{d\bar{z}}\right|^2, v \in \mathcal{J}^{(0)}_{p-d,0} \tag{29}$$

and

$$\rho^2 \text{div}(v \times v_{x_2}, v_{x_1} \times v) = \text{div}(\rho^2 v \times v_{x_2}, \rho^2 v_{x_1} \times v) - \frac{\partial\rho^2}{\partial x_1} v \times v_{x_2} - \frac{\partial\rho^2}{\partial x_2} v_{x_1} \times v \tag{30}$$

Integrating by parts and using (29) and (30), we get

$$\frac{1}{2}\int_G \rho^2 |\nabla v|^2 \, dx = \pi(p-d) - \frac{1}{2}\int_G \left(\frac{\partial\rho^2}{\partial x_1} v \times v_{x_2} + \frac{\partial\rho^2}{\partial x_2} v_{x_1} \times v \right) + 2\int_G \rho^2 \left|\frac{dv}{d\bar{z}}\right|^2 \tag{31}$$

Using (31), (22) can be rewritten as

$$F_\varepsilon[\eta(z,\xi)] = \pi(p-d) + R_\varepsilon[\eta(z,\xi)] \tag{32}$$

where $R_\varepsilon[\eta(z,\xi)] \to 0$ as $\xi \to \partial\Omega$ due to (26) and

$$\left|\frac{d\eta(z,\xi)}{d\bar{z}}\right|^2 = |\gamma(z,\xi)|^2 \left|\frac{db(z,\xi)}{d\bar{z}}\right|^2$$

Therefore, we can choose ξ sufficiently close to $\partial\Omega$ so that (25) holds. Contradiction. □

3 Model problem.

Performing the change of variables Ψ in (21) and (22) ([7], Proposition 20), we get

$$H_\varepsilon[w] = \frac{d}{2}\int_0^{2\pi}\int_{h_\varepsilon}^0 |\partial_h w|^2 \rho^4 \mathrm{d}h\mathrm{d}\theta + \\ + \frac{1}{2d}\int_0^{2\pi}\int_{h_\varepsilon}^0 (|\partial_\theta w|^2 - d^2|w|^2)\mathrm{d}h\mathrm{d}\theta + \frac{1}{4\varepsilon^2}\int_0^{2\pi}\int_{h_\varepsilon}^0 \rho^2(|w|^2-1)^2\frac{\mathrm{d}h\mathrm{d}\theta}{d|\nabla\theta|^2}. \quad (33)$$

and

$$F_\varepsilon[v] = \frac{d}{2}\int_0^{2\pi}\int_{h_\varepsilon}^0 |\partial_h v|^2 \rho^4 \mathrm{d}h\mathrm{d}\theta + \frac{1}{2d}\int_0^{2\pi}\int_{h_\varepsilon}^0 |\partial_\theta v|^2 \mathrm{d}h\mathrm{d}\theta + \\ + \frac{i}{2}\int_0^{2\pi}\int_{h_\varepsilon}^0 \left(v\overline{\partial_\theta v} - \overline{v}\partial_\theta v\right)\mathrm{d}h\mathrm{d}\theta + \frac{1}{4\varepsilon^2}\int_0^{2\pi}\int_{h_\varepsilon}^0 \rho^2(|v|^2-1)^2\frac{\mathrm{d}h\mathrm{d}\theta}{d|\nabla\theta|^2}. \quad (34)$$

where h_ε is given by (17) and, using the properties of *abdeg* ([7]), $h_\varepsilon \to h_0 := -2\pi\frac{d}{\mathrm{cap}(G)}$.

Proof for Theorem 2. As in Proposition 2, we assume $p > d$ and $q = d$, the general case is treated in Section 3.2. We argue by contradiction. Assume $\inf_{v\in\mathcal{J}^{(0)}_{p-d,0}} F_\varepsilon[v]$ is attained for some $v^*_\varepsilon \in \mathcal{J}^{(0)}_{p-d,0}$. In view of Proposition 2,

$$F_\varepsilon[v^*_\varepsilon] = \inf_{v\in\mathcal{J}^{(0)}_{p-d,0}} F_\varepsilon[v] \leq \pi(p-d) \quad (35)$$

On the other hand,

$$F_\varepsilon[v^*_\varepsilon] \geq \inf_{v\in K_\varepsilon} F_\varepsilon[v] \quad (36)$$

where

$$K_\varepsilon := \{v \in H^1(G), v = v^*_\varepsilon \text{ on } \partial\Omega, -\frac{1}{2} < \mathrm{abdeg}(v,G) < \frac{1}{2}, \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\omega\}, \quad (37)$$

since v^*_ε minimizes F_ε with respect to its own boundary condition on $\partial\Omega$, while the more restrictive condition $|v| = 1, \deg(v,\partial\omega) = 0$ on $\partial\omega$ is replaced with the Neumann boundary condition $\frac{\partial v}{\partial\nu} = 0$ on $\partial\omega$ which arises naturally from minimization and thus poses no additional restrictions. The strategy of the proof will be as follows. For sufficiently thin domains and $q = d$, we will consider a the minimizations problem for a simplified quadratic functional M, obtained essentially from the first three terms in (34). This will enable us to obtain the key estimate

$$F_\varepsilon[v^*_\varepsilon] \geq \inf_{v\in K_\varepsilon} M_\varepsilon[v] > \pi(p-d) \quad (38)$$

which leads to contradiction with Proposition 2 and thus completes the proof of Theorem 2. Finally, we will generalize this result for domains of arbitrary thickness and $q \geq d$.

3.1 Model problem in thin domains

Assume that the domain G is sufficiently thin, i.e. $\mathrm{cap}(G) \geq c_0$ for sufficiently large c_0. It follows from Lemma 1 that $\rho \to 1$ in $C^1(\bar{G})$. Thus there exists $c_\varepsilon > 0$, $c_\varepsilon \to 0$ such that

$$\rho^4 \geq e^{c_\varepsilon h}, h \in [h_\varepsilon, 0]. \tag{39}$$

Consider a simplified quadratic functional M_ε, which essentially contains the first three terms in (34):

$$M_\varepsilon[v] = M_\varepsilon[v, G] := \frac{d}{2}\int_0^{2\pi}\int_{h_\varepsilon}^0 |\partial_h v|^2 e^{c_\varepsilon h} \mathrm{d}h\mathrm{d}\theta + \frac{1}{2d}\int_0^{2\pi}\int_{h_\varepsilon}^0 |\partial_\theta v|^2 \mathrm{d}h\mathrm{d}\theta + \frac{i}{2}\int_0^{2\pi}\int_{h_\varepsilon}^0 \left(v\overline{\partial_\theta v} - \overline{v}\partial_\theta v\right) \mathrm{d}h\mathrm{d}\theta \tag{40}$$

Clearly, due to (39), $F_\varepsilon[v] \geq M_\varepsilon[v]$ for any $v \in K_\varepsilon$. The minimization problem for M_ε in K_ε plays an important role in the further analysis and is called *the model problem.*

The function $v^*_\varepsilon = v^* = v^*(h, \theta)$, defined in (35) is 2π periodic in θ. Thus we can present it as Fourier series

$$v^*(0, \theta) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ik\theta},$$

where $\alpha_k = \alpha_k(\varepsilon) \in \mathbb{C}$ are ε - dependent.

Lemma 3. *(Properties of α_k). The coefficients α_k satisfy*

(i)

$$\sum_{k=-\infty}^{\infty} k|\alpha_k|^2 = p - d; \tag{41}$$

(ii)

$$\sum_{k=-\infty}^{\infty} \alpha_k \overline{\alpha_{k+n}} = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0. \end{cases} \tag{42}$$

(iii) Up to a subsequence ε_n, still denoted ε,

$$\begin{cases} |\alpha_0| \to 1 \\ |\alpha_k| \to 0, k \neq 0 \end{cases} \quad \text{as } \varepsilon \to 0 \tag{43}$$

Proof. (i) follows from the degree formula

$$\deg(v, \partial\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega} \bar{v}\frac{\partial v}{\partial\tau}, \tag{44}$$

valid for $v \in C^1(\partial\Omega, \mathbb{S}^1)$.
In order to get (ii), observe that $v^*(0, \theta)\overline{v^*(0, \theta)} = |v^*(0, \theta)|^2 \equiv 1$. Therefore, for $n \in \mathbb{Z}$,

$$\frac{1}{2\pi}\int_0^{2\pi} v^*(0, \theta)\overline{v^*(0, \theta)}e^{in\theta}\, d\theta = \sum_{k=-\infty}^{\infty} \alpha_k\overline{\alpha_{k+n}} = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n \neq 0. \end{cases}$$

Let us show (iii). Arguing as in the proof of Theorem 4 [7], we conclude that $u_\varepsilon \rightharpoonup u_\infty$ in $H^1(G)$ where u_∞ minimizes (11). On the other hand, $u_\varepsilon = \rho e^{id\theta} v^*_\varepsilon = u^{(d)} v^*_\varepsilon$, while, by Lemma 1, $\alpha_\varepsilon u^{(d)} \to u_\infty, \varepsilon \to 0$ in $C^{1,\beta}(G)$. Furthermore, without loss of generality we may assume that up to a subsequence $a_\varepsilon \to 1 \in \mathbb{S}^1$. Thus, up to passing to a further subsequence, $v^*_\varepsilon \rightharpoonup 1$ in $H^1(G)$. By trace theorem, $v^*_\varepsilon \rightharpoonup 1$ in $H^{1/2}(\partial\Omega)$, and (43) follows. □

We look for minimizers $v(h,\theta)$ of (40) in (37) also in form of Fourier series

$$v_\varepsilon(h,\theta) = \sum_{k=-\infty}^{\infty} \alpha_k f_k^\varepsilon(h) e^{ik\theta} \tag{45}$$

where $\alpha_k \in \mathbb{C}$ and $f_k^\varepsilon \in C^\infty([h_\varepsilon, 0], \mathbb{C})$ s.t. $f_k^\varepsilon(0) = 1$. The expression for M thus becomes

$$M_\varepsilon[v] = \pi \sum_{k=-\infty}^{\infty} |\alpha_k|^2 \int_{h_\varepsilon}^0 \left[d|f_k'(h)|^2 e^{c_\varepsilon h} + \left(\frac{k^2}{d} + 2k\right) |f_k(h)|^2 \right] dh \tag{46}$$

Lemma 4. *Let*

$$m_k^\varepsilon := \inf \left\{ \int_{h_\varepsilon}^0 \left[d|f_k'(h)|^2 e^{c_\varepsilon h} + \left(\frac{k^2}{d} + 2k\right) |f_k(h)|^2 \right] dh; f_k(0) = 1, f_k'(h_\varepsilon) = 0 \right\}$$

Then

$$m_k^\varepsilon \geq \hat{m}_k + o_\varepsilon(1)$$

where

$$\hat{m}_k := \inf \left\{ \int_{h_0}^0 \left[d|f_k'(h)|^2 + \left(\frac{k^2}{d} + 2k\right) |f_k(h)|^2 \right] dh; f_k(0) = 1, f_k'(h_0) = 0 \right\} \tag{47}$$

and $h_0 := \lim_{\varepsilon \to 0} h_\varepsilon$.

Proof. Clearly

$$m_k^\varepsilon \geq \hat{m}_k^\varepsilon + o_\varepsilon(1) \tag{48}$$

where

$$\hat{m}_k^\varepsilon := \inf \left\{ \int_{h_\varepsilon}^0 \left[d|f_k'(h)|^2 e^{c_\varepsilon h} + \left(\frac{k^2}{d} + 2k\right) |f_k(h)|^2 \right] dh; f_k(0) = 1, f_k'(h_0) = 0 \right\} \tag{49}$$

Minimizing (49) and (47) leads to the following problems

$$\begin{cases} -d\left(f_k^{\varepsilon\prime}(h) e^{c_\varepsilon h}\right)' + \left(\frac{k^2}{d} + 2k\right) f_k^\varepsilon(h) = 0; \\ f_k^\varepsilon(0) = 1; \\ {f^\varepsilon}'_k(h_0) = 0. \end{cases} \tag{50}$$

and

$$\begin{cases} -d f_k''(h) + \left(\frac{k^2}{d} + 2k\right) f_k(h) = 0; \\ f_k(0) = 1; \\ f_k'(h_0) = 0. \end{cases} \tag{51}$$

correspondingly. Furthermore, multiplying (50) and (51) by f_k^ε and f_k respectively and integrating by parts, we get

$$\hat{m}_k^\varepsilon = \int_{h_0}^0 \left[d|f_k'(h)|^2 e^{c_\varepsilon h} + \left(\frac{k^2}{d} + 2k\right) |f_k(h)|^2 \right] dh = d f_k^{\varepsilon\prime}(0) f_k(0) = d f_k^{\varepsilon\prime}(0) \tag{52}$$

and similarly

$$\hat{m}_k = d{f_k}'(0) \tag{53}$$

Denoting $f_k^{\varepsilon\prime} = g_k^\varepsilon$, the equation (50) becomes

$$\begin{cases} f_k^{\varepsilon\prime} = g_k^\varepsilon; \\ d g_k^{\varepsilon\prime} = -d g_k^\varepsilon + \left(\frac{k^2}{d} + 2k\right) e^{-c_\varepsilon h} f_k^\varepsilon(h). \end{cases} \tag{54}$$

By Theorem 7.2 [2], the solutions of the regularly perturbed system (54) converge to the solutions of the limiting system

$$\begin{cases} f_k' = g_k; \\ dg_k{}' = -dg_k + \left(\frac{k^2}{d} + 2k\right) f_k(h). \end{cases}$$

uniformly on $[h_0, 0]$. In view of (52), (53), $\hat{m}_k^\varepsilon = \hat{m}_k + o_\varepsilon(1)$, which, in conjunction with (48), concludes Lemma 4.

□

Lemma 5. *Assume α_k and f_k are given by (45). Then $\exists h_0^* < 0$ s.t. for any $h_0^* < h_0 < 0$ and for sufficiently small ε we have*

$$\pi d \sum_{k=-\infty}^{\infty} f_k'(0)|\alpha_k|^2 > \pi \sum_{k=-\infty}^{\infty} k|\alpha_k|^2 \tag{55}$$

Lemma 5 plays the crucial role in the proof of Theorem 2. In view of Lemma 4, (52) and the degree formula (41), the inequality (55) implies (38) and thus completes the proof of Theorem 2 for $q = d$ in thin domains.

Proof. (Lemma 5) We start with solving (50) in three different cases depending on the relation between k and d:

Case I. $k = 0$ *or* $k = -2d$.
Trivial constant solution $f_k(h) \equiv 1$.

Case II. $k \neq 0, -1, \cdots, -2d$.

$$f_k(h) = \frac{e^{-2\lambda_2 h_0} e^{\lambda_2 h} + e^{-\lambda_2 h}}{1 + e^{-2\lambda_2 h_0}}, \quad \lambda_2 = \sqrt{\frac{k^2}{d^2} + \frac{2k}{d}} \tag{56}$$

Furthermore,

$$df_k'(0) = \sqrt{k^2 + 2kd} \left(\frac{1 - e^{2\lambda_2 h_0}}{1 + e^{2\lambda_2 h_0}} \right) \tag{57}$$

Case III. $k = -1, \cdots, -2d + 1$.

$$f_k(h) = \tan \lambda_3 h_0 \sin \lambda_3 h + \cos \lambda_3 h, \quad \lambda_3 = \sqrt{-\frac{k^2}{d^2} - \frac{2k}{d}} \tag{58}$$

Similarly to the previous case, we have

$$df_k'(0) = \sqrt{-k^2 - 2kd} \tan(h_0 \lambda_3). \tag{59}$$

The following Lemma plays an important role in the proof of Lemma 5.

Lemma 6. *Fix an arbitrary natural number $k_0 \geq 2d$. Then for any $n = 1, \ldots, k_0$ there exist two constants $C = C(k_0, n) > 0$ and $c^\varepsilon > 0$ such that $c^\varepsilon \to 1$ as $\varepsilon \to 0$ and for sufficiently small ε*

$$|\alpha_{-n}| \leq c^\varepsilon |\alpha_n| + C \sum_{k \leq -k_0, k \geq 1} |\alpha_k|^2. \tag{60}$$

Proof. It follows from (42) with $n = 1$ that

$$|\alpha_{-1}||\alpha_0| \leq |\alpha_1||\alpha_0| + \sum_{|k|\geq 1} |\alpha_k|^2. \tag{61}$$

Rewriting (61) as

$$|\alpha_{-1}|(|\alpha_0| - |\alpha_1|) \leq |\alpha_1||\alpha_0| + \sum_{k\geq 1, k\leq -2} |\alpha_k|^2$$

and using the convergence (43) we conclude that

$$|\alpha_{-1}| \leq c_\varepsilon^1|\alpha_1| + C_1 \sum_{k\geq 1, k\leq -2} |\alpha_k|^2 \tag{62}$$

with $c_\varepsilon^1 \to 1$ as $\varepsilon \to 0$ and C_1 independent of ε.
We proceed with using the orthogonality property (42) for $n = 2$, which yields

$$|\alpha_{-2}|(|\alpha_0| - |\alpha_{-4}|) \leq |\alpha_2||\alpha_0| + |\alpha_{-1}|^2 + \tilde{C}_2 \sum_{k\geq 1, k\leq -3} |\alpha_k|^2 \tag{63}$$

Applying the estimate (62) and making use of the convergence properties of coefficients (43), we obtain

$$|\alpha_{-2}| \leq c_\varepsilon^2|\alpha_2| + C_2 \sum_{k\geq 1, k\leq -3} |\alpha_k|^2 \tag{64}$$

with $c_\varepsilon^2 \to 1$ as $\varepsilon \to 0$ and C_2 independent of ε. The estimate (68) allows us to improve the estimate (62) to

$$|\alpha_{-1}| \leq c_\varepsilon^1|\alpha_1| + C_1 \sum_{k\geq 1, k\leq -3} |\alpha_k|^2 \tag{65}$$

(with possibly different C_1). Now, once more we make use of the orthogonality property (42) for $n = 3$, which, together with the estimates (68) and (67), yields

$$|\alpha_{-3}| \leq c_\varepsilon^3|\alpha_3| + C_3 \sum_{k\geq 1, k\leq -4} |\alpha_k|^2 \tag{66}$$

where $c_\varepsilon^3 \to 1$ as $\varepsilon \to 1$ and C_3 independent of ε. The estimate (66), in turn, may be used to improve the estimates (67) and (68):

$$|\alpha_{-1}| \leq c_\varepsilon^1|\alpha_1| + C_1 \sum_{k\geq 1, k\leq -4} |\alpha_k|^2 \tag{67}$$

and

$$|\alpha_{-2}| \leq c_\varepsilon^2|\alpha_2| + C_2 \sum_{k\geq 1, k\leq -4} |\alpha_k|^2. \tag{68}$$

Repeating the above procedure k_0 times, we arrive at the estimate (60). □

We now return to the proof of Lemma 5. It follows from (68) that one can find some $h_0^* < 0$ such that for $h_0 \in (h_0^*, 0)$ we have

$$df_k^{'}(0) - k \geq 0, \forall k \geq 1. \tag{69}$$

For large values of k the inequality (69) can be improved, namely, $\exists k_0 \geq 2d$, depending only on h_0 and d, s.t. for $|k| \geq k_0$

$$df_k^{'}(0) - k \geq \frac{d}{4} \tag{70}$$

The inequality (70) trivially holds for $k \le -k_0$, while for $k \ge k_0$ is follows from the elementary inequality $\sqrt{k^2+2kd} \ge k(1+\frac{d}{2k})$, which is true for sufficiently large k. Introduce

$$I_1 := \sum_{k=1}^{k_0} |\alpha_k|^2$$

and

$$I_2 := \sum_{|k|>k_0} |\alpha_k|^2.$$

In view of (70)

$$\sum_{k=-\infty}^{\infty} (df_k^{'}(0)-k)|\alpha_k|^2 \ge \sum_{|k|\le k_0} (df_k^{'}(0)-k)|\alpha_k|^2 + \frac{d}{4}I_2. \tag{71}$$

Applying Lemma 6 and (69), we obtain the following estimate of the first sum in (71)

$$\sum_{|k|\le k_0} (df_k^{'}(0)-k)|\alpha_k|^2 = \sum_{k=1}^{k_0}[(df_k^{'}(0)-k)|\alpha_k|^2 + (df_{-k}^{'}(0)+k)|\alpha_{-k}|^2] =$$

$$= \frac{1}{2d+1}\sum_{k=1}^{k_0} df_k^{'}(0)|\alpha_k|^2 + \sum_{k=1}^{k_0}[\frac{2d^2}{2d+1}f_k^{'}(0)-k)|\alpha_k|^2 + (df_{-k}^{'}(0)+k)|\alpha_{-k}|^2] \ge$$

$$\ge \frac{1}{2d+1}I_1 + \sum_{k=1}^{k_0}[(\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0))|\alpha_{-k}|^2] + I_3 \tag{72}$$

where, due to (43),

$$I_3 := C_1\sum_{k=1}^{k_0}|\alpha_{-k}| \sum_{m\ge 1, m\le -k_0} |\alpha_m|^2 + C_2\left(\sum_{m\ge 1, m\le -k_0} |\alpha_m|^2\right)^2 = o(I_1+I_2), \varepsilon \to 0. \tag{73}$$

It remains to obtain a lower bound for $\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0)$ for $1 \le k \le k_0$. Clearly, $\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0) > 0$ for $k \ge 2d$. Moreover, the explicit expressions (57) and (59) yield that for all $1 \le k \le 2d-1$

$$\frac{\partial}{\partial h_0}\left(\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0)\right)\bigg|_{h_0=0} = \frac{1}{d}\left[\frac{2d}{2d+1}(-k^2-2kd) + (2kd-k^2)\right] < 0$$

The latter inequality guarantees that for $0 > h_0 > h_0^*$ sufficiently small

$$\sum_{k=1}^{k_0}[(\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0))|\alpha_{-k}|^2] \ge 0 \tag{74}$$

Finally, it follows from (71), (72), (73) and (74) that for $0 > h_0 > h_0^*$

$$\sum_{k=-\infty}^{\infty} (df_k^{'}(0)-k)|\alpha_k|^2 \ge \frac{1}{2d+1}I_1 + \frac{d}{4}I_2 + \sum_{k=1}^{k_0}[(\frac{2d^2}{2d+1}f_k^{'}(0) + df_{-k}^{'}(0))|\alpha_{-k}|^2] + o(I_1+I_2) > 0$$

for sufficiently small ε. This completes the proof of Lemma 5. □

3.2 Model problem in general case.

Let $p > d$, $q \geq d$ and the capacity of the domain G is arbitrary.

Assume $\inf_{v\in\mathcal{J}^{(0)}_{p-d,q-d}} F_\varepsilon[v]$ is attained for some $v^*_\varepsilon \in \mathcal{J}^{(0)}_{p-d,q-d}$. In view of Proposition 2,

$$F_\varepsilon[v^*_\varepsilon] = \inf_{v\in\mathcal{J}^{(0)}_{p-d,q-d}} F_\varepsilon[v] \leq \pi(|p-d| + |q-d|) \tag{75}$$

Consider the minimization problem for the functional

$$W_\varepsilon[v] = W_\varepsilon[v,G] := \frac{d}{2}\int_0^{2\pi}\int_{h_0}^0 |\partial_h v|^2 \mathrm{d}h\mathrm{d}\theta + \frac{1}{2d}\int_0^{2\pi}\int_{h_0}^0 |\partial_\theta v|^2 \mathrm{d}h\mathrm{d}\theta + \\ + \frac{i}{2}\int_0^{2\pi}\int_{h_0}^0 \left(v\overline{\partial_\theta v} - \overline{v}\partial_\theta v\right)\mathrm{d}h\mathrm{d}\theta + \frac{1}{4\varepsilon^2}\int_0^{2\pi}\int_{h_0}^0 \rho^2(|v|^2-1)^2 \frac{\mathrm{d}h\mathrm{d}\theta}{d|\nabla\theta|^2} \tag{76}$$

in the class $\mathcal{J}^{(0)}_{p-d,q-d}(G)$, $h_0 = -\frac{2\pi d}{\mathrm{cap}(G)}$.

(i) If $h_0 > 2h^*_0$, consider a closed simple curve $\mathcal{L}$ which lies within G so that $\omega \subset \mathrm{int}(G)$. Then $G = G_1 \cup G_2$, where $G_1 := \Omega \setminus \mathrm{int}(\mathcal{L})$ and $G_2 := \mathrm{int}(\mathcal{L}) \setminus \omega$. Moreover, we can always choose $\mathcal{L}$ so that $h^1 := -\frac{2\pi d}{\mathrm{cap}(G_1)} > h^*_0$ and $h^2 := -\frac{2\pi d}{\mathrm{cap}(G_2)} > h^*_0$. Then

$$\inf_{v\in\mathcal{J}^{(0)}_{p-d,0}} W_\varepsilon[v,G] \geq \inf_{v_1\in K_1} M[v_1,G_1] + \inf_{v_2\in K_2} M[v_2,G_2]$$

where $M[v_i,G_i], i = 1,2$ is given by (40) and

$$K_1 := \{u \in H^1(G,\mathbb{C}), |u| = 1 \text{ on } \partial\Omega, \deg(u,\partial\Omega) = p-d, -\frac{1}{4} < \mathrm{abdeg}(u,G_1) < \frac{1}{4}, \frac{\partial u}{\partial\nu} = 0 \text{ on } \mathcal{L}\}.$$

$$K_2 := \{u \in H^1(G,\mathbb{C}), |u| = 1 \text{ on } \partial\omega, \deg(u,\partial\omega) = q-d, -\frac{1}{4} < \mathrm{abdeg}(u,G_2) < \frac{1}{4}, \frac{\partial u}{\partial\nu} = 0 \text{ on } \mathcal{L}\}.$$

Consequently, by the main result of Section 3.1, $\inf_{v\in\mathcal{J}^{(0)}_{p-d,q-d}} W_\varepsilon[v,G] > \pi(|p-d| + |q-d|)$.

(ii) If $2h^*_0 > h_0$, split G into three disjoint domains $G = G_1 \cup G_2 \cup G_3$. The domain $G_1 = \Omega\setminus\mathrm{int}(\mathcal{L}_1)$, where $\mathcal{L}_1$ is a closed simple curve which lies within G and is such that $h^{(1)}_0 := -\frac{2\pi d}{\mathrm{cap}(G_1)} > h^*_0$. Similarly, we can choose another closed simple curve $\mathcal{L}_2$ and define $G_2 := \mathrm{int}(\mathcal{L}_2) \setminus \omega$ so that $h^{(2)}_0 := -\frac{2\pi d}{\mathrm{cap}(G_2)} > h^*_0$. Finally, set $G_3 := G \setminus (G_1 \cup G_2)$. By contradiction, assume there exists $v^* \in \mathcal{J}^{(0)}_{p-d,0}(G)$ s.t. $W_\varepsilon[v^*,G] < \pi(p-d)$. On the other hand, arguing as in (i) we can obtain the following lower bound on $W_\varepsilon[v^*,G]$:

$$W_\varepsilon[v^*,G] = \sum_{i=1}^{3} W_\varepsilon[v^*,G_i] > \pi(|p-d| + |q-d|) + W_\varepsilon[v^*,G_3]$$

Thus, in order to obtain contradiction, it suffices to establish that

$$W_\varepsilon[v^*,G_3] = \frac{d}{2}\int_0^{2\pi}\int_{h_0-h^{(1)}_0}^{h^{(2)}_0} |\partial_h v^*|^2 \mathrm{d}h\mathrm{d}\theta + \frac{1}{2d}\int_0^{2\pi}\int_{h_0-h^{(1)}_0}^{h^{(2)}_0} |\partial_\theta v^*|^2 \mathrm{d}h\mathrm{d}\theta + \\ + \frac{i}{2}\int_0^{2\pi}\int_{h_0-h^{(1)}_0}^{h^{(2)}_0} \left(v^*\overline{\partial_\theta v^*} - \overline{v^*}\partial_\theta v^*\right)\mathrm{d}h\mathrm{d}\theta + \frac{1}{4\varepsilon^2}\int_0^{2\pi}\int_{h_0-h^{(1)}_0}^{h^{(2)}_0} \rho^2(|v^*|^2-1)^2 \frac{\mathrm{d}h\mathrm{d}\theta}{d|\nabla\theta|^2} \geq 0 \tag{77}$$

Using Lemma 6 [5] we may conclude that v^* is vortexless away from the boundary ∂G, in particular, in G_3. Moreover,

$$\deg(\frac{v^*}{|v^*|},\mathcal{L}_1) = \deg(\frac{v^*}{|v^*|},\mathcal{L}_2) = 0 \tag{78}$$

Hence, writing $v^* = |v^*|e^{i\psi}$, we get

$$\frac{i}{2}\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \left(v^*\overline{\partial_\theta v^*} - \overline{v^*}\partial_\theta v^*\right) = \frac{1}{2}\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} |v^*|^2\frac{d\psi}{d\theta} = \frac{1}{2}\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} (|v^*|^2-1)\frac{d\psi}{d\theta} \quad (79)$$

since $\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \frac{d\psi}{d\theta} = 0$ due to (78). By Cauchy-Schwartz

$$\left|\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \left(v^*\overline{\partial_\theta v^*} - \overline{v^*}\partial_\theta v^*\right)\right| \leq \delta\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \left|\frac{d\psi}{d\theta}\right|^2 + \frac{1}{\delta}\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} (|v^*|^2-1)^2$$

for any $\delta > 0$. Thus,

$$W_\varepsilon[v^*, G_3] \geq \frac{d}{2}\int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} |\partial_h v^*|^2 \mathrm{d}h\mathrm{d}\theta + \int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \left(\frac{1}{2d} - 2\delta\right)|\partial_\theta v^*|^2 \mathrm{d}h\mathrm{d}\theta +$$
$$+ \int_0^{2\pi}\int_{h_0-h_0^{(1)}}^{h_0^{(2)}} \left(\frac{1}{4\varepsilon^2}\frac{\rho^2}{d|\nabla\theta|^2} - \frac{1}{2\delta}\right)(|v^*|^2-1)^2 \mathrm{d}h\mathrm{d}\theta \geq 0 \quad (80)$$

for sufficiently small δ. Thus, we showed that, for sufficiently small ε, $W_\varepsilon[v, G] > \pi(|p-d| + |q-d|)$ for any $v \in \mathcal{J}^{(0)}_{p-d,q-d}$. Similarly to Section 3.1 (i.e. Lemma 4) we can show that the same is true for F_ε, i.e. $F_\varepsilon[v^*_\varepsilon] > \pi(|p-d| + |q-d|)$. The obtained inequality contradicts Proposition 2 and concludes the proof of Theorem 2.

4 Proof of Theorem 3.

In this section we consider the minimization problem for (1) in $\mathcal{J}^{(0)}_{p,q}$ for an arbitrary integers p and q. The approach to proving the Theorem 3 is essentially different from the one used in the proof of Theorem 2. The reason that the decomposition (20) is no longer valid. Instead, we will follow closely the strategy, described in ([3]). We start with the following preliminary results.

Proposition 3. *For any $\varepsilon > 0$,*

$$m_{p,q} := \inf_{u\in\mathcal{J}^{(0)}_{p,q}} E_\varepsilon[u] \leq \pi(|p| + |q|)$$

Proposition 4. *Let u_ε be a solution of Ginzburg-Landau equation* (2) *such that $E_\varepsilon[u_\varepsilon] < \pi(|p| + |q|) + e^{-1/\varepsilon}$. Then there exists $\gamma_\varepsilon = \text{const} \in S^1$ such that for any compact set K in G*

$$\|u_\varepsilon - \gamma_\varepsilon\|_{C^l(K)} = o(\varepsilon^m), \;\; \textit{as } \varepsilon \to 0, \forall m > 0, l \in \mathbb{N} \quad (81)$$

$$\int_G (|u_\varepsilon|^2 - 1)^2\, dx = o(\varepsilon^m), \;\; \textit{as } \varepsilon \to 0, \forall m > 0. \quad (82)$$

The propositions 3 and 4 were proved for $p = q = 1$ in [5]. These proofs can be modified for arbitrary integers p and q in a straightforward way.

Proof. (Theorem 3) By contradiction, assume that $\inf_{u\in\mathcal{J}^{(0)}_{p,q}} E_\varepsilon[u] = E_\varepsilon[u_\varepsilon]$ for some $u_\varepsilon \in \mathcal{J}^{(0)}_{p,q}$. We are going to construct a lower bound on $E_\varepsilon[u_\varepsilon]$, which would contradict Proposition 3. We will follow closely the strategy of [3]. For convenience, the proof will be split into several steps.
Step 1. Conformal equivalence to a circular annulus.

Proposition 5. *[3] Suppose G is such that $m_{p,q}$ is attained. Then the same holds for $B = B_{R_1}(0) \setminus B_{R_2}(0)$ with R_1 and R_2 satisfying* $\operatorname{cap}(G) = \frac{2\pi}{\ln(R_1/R_2)}$.

Step 2. Construction of the model problem. Let $R_m = \frac{R_1 - R_2}{2}$. We split the domain B as $B = B_{ext} \cup B_{int}$, where $B_{ext} := B_{R_1}(0) \setminus B_{R_m}(0)$, $B_{int} := B_{R_m}(0) \setminus B_{R_2}(0)$. Following [3], we can obtain the lower bound in B_{ext}:

$$E_\varepsilon[u_\varepsilon, B_{ext}] \geq \inf_{w \in K^1_\varepsilon} S^{(1)}_\varepsilon[w] \tag{83}$$

where

$$K^1_\varepsilon = \{w \in H^1(B_{ext}), w = u_\varepsilon \text{ on } \partial B_{R_1}(0), \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial B_{R_m}(0)\} \tag{84}$$

and

$$S^{(1)}_\varepsilon[w] := \frac{1}{2}\int_{R_m}^{R_1} dr \int_0^{2\pi} |\nabla w|^2 \, d\phi + \int_{R_m}^{\rho_1} dr \int_0^{2\pi} d\phi \left(\frac{1}{2\varepsilon^2}(Re(w) - 1)^2 - 2\varepsilon^2 (Im(w))^2\right) \tag{85}$$

for some $\rho_1 \in (R_m, R_1)$. A similar lower bound holds in B_{int}:

$$E_\varepsilon[u_\varepsilon, B_{int}] \geq \inf_{w \in K^2_\varepsilon} S^{(2)}_\varepsilon[w] \tag{86}$$

where

$$K^2_\varepsilon = \{w \in H^1(B_{int}), w = u_\varepsilon \text{ on } \partial B_{R_2}(0), \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial B_{R_m}(0)\}$$

and

$$S^{(2)}_\varepsilon[w] := \frac{1}{2}\int_{R_2}^{R_m} dr \int_0^{2\pi} |\nabla w|^2 \, d\phi + \int_{\rho_2}^{R_m} dr \int_0^{2\pi} d\phi \left(\frac{1}{2\varepsilon^2}(Re(w) - 1)^2 - 2\varepsilon^2 (Im(w))^2\right)$$

for some $\rho_1 \in (R_2, R_m)$.

Step 3 (Analysis of the model problem). We proceed with analyzing the model problem (84) - (85). Since $S_\varepsilon[w] \equiv S_\varepsilon[\overline{w}]$, without loss of generality we may assume $p > 0$ (the case $p < 0$ may be addressed in the same way by taking the complex conjugation). We may expand 2π - periodic minimizer u_ε at $r = R_1$ as

$$u_\varepsilon(R_1, \phi) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\phi + b_n \sin n\phi)$$

It follows from the degree formula (44) that

$$p = \sum_{n=1}^{\infty} n[Re(a_n)Im(b_n) - Re(b_n)Im(a_n)] \tag{87}$$

Let w be the minimizer of (85) in (84). Then w satisfies

$$\begin{cases} -\Delta Re(w) + \frac{1}{\varepsilon^2}V(r)(Re(w) - 1) = 0, R_1 < r < R_m, \\ -\Delta Im(w) - \varepsilon^2 V(r) Im(w) = 0, R_1 < r < R_m, \\ w(r, \phi) = w(r, \phi + 2\pi), \\ w = u_\varepsilon \text{ for } r = R_1; \frac{\partial w}{\partial r} = 0 \text{ for } r = R_m. \end{cases} \tag{88}$$

Here $V(r) = 1$ when $r \in (R_m, \rho_1)$ and $V(r) = 0$ otherwise. We look for solutions of (88) in the form

$$w_\varepsilon(r, \phi) = 1 + (a_0 - 1)w^{(1)}_0(r) + \sum_{n=1}^{\infty} w^{(1)}_n(r)(Re(a_n)\cos n\phi + Re(b_n)\sin n\phi)$$

$$+i\sum_{n=1}^{\infty} w_n^{(2)}(r)(Im(a_n)\cos n\phi + Im(b_n)\sin n\phi)$$

In a similar way to ([3]) we conclude that $a_0 \in \mathbb{R}$ and that

$$S_\varepsilon^{(1)}[w_\varepsilon] = P_0 + \pi\sum_{n=1}^{\infty} n[P_n(|Re(a_n)|^2 + |Re(b_n)|^2) + Q_n(|Im(a_n)|^2 + |Im(b_n)|^2)] \tag{89}$$

Here $P_0 \geq 0$, $nP_n = \frac{d}{dr}w_n^{(1)}(R_1)$ and $nQ_n = \frac{d}{dr}w_n^{(2)}(R_1)$. The functions $w_n^{(1)}$ and $w_n^{(2)}$ satisfy

$$\begin{cases} -\frac{d^2}{dr^2}w_n^{(1)}(r) + (n^2 + \frac{1}{\varepsilon^2}V(r))w_n^{(1)}(r) = 0, R_m < r < R_1; \\ w_n^{(1)}(R_1) = 1; \\ \frac{d}{dr}w_n^{(1)}(R_m) = 0. \end{cases} \tag{90}$$

and

$$\begin{cases} -\frac{d^2}{dr^2}w_n^{(2)}(r) + (n^2 - \varepsilon^2 V(r))w_n^{(2)}(r) = 0, R_m < r < R_1; \\ w_n^{(2)}(R_1) = 1; \\ \frac{d}{dr}w_n^{(2)}(R_m) = 0. \end{cases} \tag{91}$$

Solving (90) and (91), denoting $\rho := R_1 - \rho_1$ and $h := \frac{R_2 - R_1}{2}$, we obtain

$$P_n = 1 - \frac{2e^{-2n\rho}\alpha_n^{(1)}}{\beta_n^{(1)}}$$

where

$$\alpha_n^{(1)} = e^{-\rho\sqrt{n^2+\frac{1}{\varepsilon^2}}}\left(n - \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right) + e^{(-2h+\rho)\sqrt{n^2+\frac{1}{\varepsilon^2}}\rho}\left(n + \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right)$$

and

$$\beta_n^{(1)} = e^{-\rho\sqrt{n^2+\frac{1}{\varepsilon^2}}}\left(n + \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right) + e^{(-2h+\rho)\sqrt{n^2+\frac{1}{\varepsilon^2}}\rho}\left(n - \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right)$$

$$+e^{-\rho\left(\sqrt{n^2+\frac{1}{\varepsilon^2}}+2n\right)}\left(n - \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right) + e^{(-2h+\rho)\sqrt{n^2+\frac{1}{\varepsilon^2}}-2n\rho}\left(n + \sqrt{n^2 + \frac{1}{\varepsilon^2}}\right)$$

Similarly,

$$Q_n = 1 - \frac{2e^{-2n\rho}\alpha_n^{(2)}}{\beta_n^{(2)}}$$

where

$$\alpha_n^{(2)} = e^{-\rho\sqrt{n^2-\varepsilon^2}}\left(n - \sqrt{n^2 - \varepsilon^2}\right) + e^{(-2h+\rho)\sqrt{n^2-\varepsilon^2}\rho}\left(n + \sqrt{n^2 - \varepsilon^2}\right)$$

and

$$\beta_n^{(2)} = e^{-\rho\sqrt{n^2-\varepsilon^2}}\left(n + \sqrt{n^2 - \varepsilon^2}\right) + e^{(-2h+\rho)\sqrt{n^2-\varepsilon^2}\rho}\left(n - \sqrt{n^2 - \varepsilon^2}\right)$$

$$+e^{-\rho\left(\sqrt{n^2-\varepsilon^2}+2n\right)}\left(n - \sqrt{n^2 - \varepsilon^2}\right) + e^{(-2h+\rho)\sqrt{n^2-\varepsilon^2}-2n\rho}\left(n + \sqrt{n^2 - \varepsilon^2}\right)$$

As $\varepsilon \to 0$, we get that for any fixed $n \geq 1$

$$P_n \to 1 + 2e^{-2n\rho} \text{ and } Q_n \to \frac{1 - e^{-2nh}}{1 + e^{-2nh}}.$$

Choose R_1 and R_2 so that $h = \frac{R_1 - R_2}{2} > \frac{\ln 7}{2}$ and $\rho = R_1 - \rho_1 < \frac{\ln 2}{2}$. With this choice of parameters we get $P_n > 2$ and $Q_n > \frac{3}{4}$ for all $n \geq 1$ for sufficiently small ε. Therefore, for sufficiently small

ε, $P_nQ_n > 1$ for all $n \geq 1$. Finally, using $P_0 \geq 0$ and the elementary inequality $a^2 + b^2 \geq 2ab$, it follows from (89) and (87) that

$$E_\varepsilon[u_\varepsilon, B_{ext}] \geq S_\varepsilon^{(1)}[w_\varepsilon] \geq \pi \sum_{n=1}^{\infty} n\sqrt{P_nQ_n}[|Re(a_n)||Im(b_n)| + |Re(b_n)||Im(a_n)|] > \pi|p| \tag{92}$$

Analogously, we can obtain a similar bound in B_{int}:

$$E_\varepsilon[u_\varepsilon, B_{int}] \geq \inf_{w \in K_\varepsilon^2} S_\varepsilon^{(2)}[w] > \pi|q| \tag{93}$$

Combining (92) and (93), we see that $E_\varepsilon[u_\varepsilon] > \pi(|p| + |q|)$, which contradicts Proposition 3. □

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